1. Course Outline

The qualitative features of the arithmetic of curves is strongly governed by geometry. Elliptic curves form a fascinating class of varieties to study because they are varieties “of intermediate type”, i.e., they are neither (geometrically) birational to $\mathbb{P}^1$, nor are they varieties of general type. K3 surfaces occupy a similar place in the theory of surfaces. This class of surfaces includes double covers of $\mathbb{P}^2$ ramified over a sextic plane curve, quartic surfaces in $\mathbb{P}^3$, and complete intersections of three quadrics in $\mathbb{P}^5$. The last fifteen years have seen a surge of activity on the arithmetic of K3 surfaces. The goal of this course is to survey some of these developments, with an emphasis on explicit methods and examples.

**Geometry of K3 surfaces.** We will start with a crash course (light on proofs) on the geometry of K3 surfaces: topological properties, including the lattice structure of $H^2(X, \mathbb{Z})$ and simple connectivity; the period point of K3 surface, the Torelli theorem and surjectivity of the period map. Good references for this material include [BHPVdV04, Ch. VIII] and [LP80].

**Potential Density.** A variety $X$ over a number field $k$ is said to satisfy potential density if there is a finite extension $L/k$ such that $X(L)$ is Zariski dense in $X$. After a quick survey of some known results for several classes of varieties, we will explain work of Bogomolov and Tschinkel that shows that K3 surfaces $X$ endowed with an elliptic fibration or with an infinite automorphism group satisfy potential density [BT98, BT99, BT00, Has03].

**Picard groups.** It is known that over a number field $k$, the (geometric) Picard group $\text{Pic}(\overline{X})$ of a projective K3 surface $X$ is a free $\mathbb{Z}$-module of rank $1 \leq \rho(\overline{X}) \leq 20$. Determining $\rho(\overline{X})$ for a given K3 surface is a difficult task; we will explain how work of van Luijk, Kloosterman, Elsenhans-Jahnel and Charles [vL07, Klo07, EJ11, Cha14] solves this problem.

**Brauer Groups.** The Galois module structure of $\text{Pic}(\overline{X})$ allows one to compute an important piece of the Brauer group $\text{Br}(X) = H^2(X_{\text{et}}, \mathbb{G}_m)$ of a locally solvable K3 surface $X$, consisting of the classes of $\text{Br}(X)$ that are killed by passage to an algebraic closure (modulo Brauer classes coming from the ground field). These classes can be used to construct counter-examples to the Hasse principle on K3 surfaces via Brauer-Manin obstructions, a topic which will dovetail with Viray’s course.

For surfaces of negative Kodaira dimension (e.g., cubic surfaces), we have $\text{Br}_1(X) = \text{Br}(X)$, so the algebraic Brauer group already gives all the information needed to determine...
Brauer-Manin obstructions to the Hasse principle and weak approximation. In contrast, for a K3 surface $X$, we know that $\text{Br}(X(\mathbb{C})) \cong (\mathbb{Q}/\mathbb{Z})^{22-\rho}$. However, a remarkable theorem of Skorobogatov and Zarhin [SZ08] says that over a number field the quotient $\text{Br}(X)/\text{Br}(k)$ is finite! The remainder of the course will be devoted to ongoing work by several authors on the computation of the non-algebraic Brauer classes on K3 surfaces, and their impact on the arithmetic of such surfaces [HVAV11,HVA13,MSTVA14].

2. Project Description

2.1. Diagonal K3 surfaces of degree 2. The goal of this project is to understand the geometric Picard group, as a Galois module, of certain double covers of $\mathbb{P}^2$ ramified along a sextic. More concretely, over a number field $k$, we want to study the hypersurface in the weighted projective space $\mathbb{P}(1,1,1,3) = \text{Proj} \ k[\![x,y,z,w]\!]$ given by

$$X_{A,B,C}/k : \quad w^2 = Ax^6 + By^6 + Cz^6$$

for some $A, B$ and $C \in k^\times$.

1. What is the rank of $\text{Pic}(\mathbb{X}_{A,B,C})$? Note that to compute this number we may assume that $A = B = C = 1$. What upper bounds are suggested by reduction modulo 3 and point counting?

2. The double cover map $\pi : X_{A,B,C} \to \mathbb{P}^2_k = \text{Proj} \ k[x,y,z]$ gives us a large supply of divisors on $X_{A,B,C}$, namely, the components of the pullback of a line in $\mathbb{P}^2$ tritangent to the branch curve $Ax^6 + By^6 + Cz^6 = 0$. What is the rank of the sublattice of $\text{Pic}(\mathbb{X}_{A,B,C})$ generated by these divisors? Does it equal $\rho(\mathbb{X}_{A,B,C})$? If so, is the sublattice saturated, i.e., is it all of the Picard group? If not, what are the missing divisor classes?

3. What is the Galois module structure of $\text{Pic}(\mathbb{X}_{A,B,C})$? The answer should depend on $A, B$ and $C$. What is the group $H^1(\text{Gal}(\overline{k}/k), \text{Pic}(\mathbb{X}_{A,B,C}))$?

4. The Hochschild-Serre spectral sequence gives rise to an isomorphism

$$\text{Br}_1(X_{A,B,C})/\text{Br}_0(X_{A,B,C}) \cong H^1(\text{Gal}(\overline{k}/k), \text{Pic}(\mathbb{X}_{A,B,C})),$$

where $\text{Br}_1(X_{A,B,C}) = \ker(\text{Br}(X_{A,B,C}) \to \text{Br}(\mathbb{X}_{A,B,C}))$ is the algebraic Brauer group, and $\text{Br}_0(X_{A,B,C}) = \text{im}(\text{Br}(k) \to \text{Br}(X_{A,B,C}))$ is the subgroup of constant algebras. Can you invert this map and produce central simple algebras over the function field $k(X_{A,B,C})$ that represent nonconstant algebraic classes in $\text{Br}(X_{A,B,C})$? Can you use these classes to give examples of Brauer-Manin obstructions to weak approximation or the Hasse principle? The paper [VA08 §3] could be of help here.

5. Specialize to $k = \mathbb{Q}$. Look at the “box”

$$\mathcal{B} := \{(A, B, C) \in \mathbb{Z}^3 : |A|, |B|, |C| \leq 100\}.$$

For which $(A, B, C) \in \mathcal{B}$ is there an algebraic obstruction to the Hasse principle on $X_{A,B,C}$? If there is no obstruction, can you find a rational point on $X_{A,B,C}$?

6. Can you construct a cubic fourfold containing a plane having $X$ as its associated K3 surface? See [HVAV11] for details on this construction. If so, can you construct a transcendental element of $\text{Br}(X)[2]$ as a quaternion algebra over the function field...
$k(X)$? How about transcendental elements in $\text{Br}(X)[2]$ arising from K3 surfaces of degree 8? See [MSTVA14] for the geometry involved here.

2.2. Twisted derived equivalence and rational points. The goal of this project is to explore a recent question coming out of work of Hassett and Tschinkel. FYI: You don’t have to know much about twisted derived categories to work on this project! However, a good understanding of the paper [HVA13] would be most helpful.

Question 2.1. Let $X$ and $Y$ be locally solvable K3 surfaces over a number field, and suppose there is an equivalence of twisted derived categories $D^b(X, \alpha) \cong D^b(Y, \beta)$ for some $\alpha \in \text{Br}(X)$ and $\beta \in \text{Br}(Y)$. Assume that $\alpha$ obstructs the Hasse principle on $X$. Is $Y(k) = \emptyset$?

Here is a concrete instance where we can explore this problem: Let $W$ be a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ ramified along a type $(2, 2)$ divisor. The two projections $\pi_i: Y \to \mathbb{P}^2$ $(i = 1, 2)$ give quadric bundle fibrations, and the degeneracy locus of this fibration is a plane sextic in $\mathbb{P}^2$. Taking the double cover of $\mathbb{P}^2$ ramified along the branch locus of $\pi_i$ gives a K3 surface. We thus obtain two K3 surfaces $X$ and $Y$ out of $W$. In [HVA13] we explain how to use $W$ to construct elements $\alpha \in \text{Br}(X)[2]$ and $\beta \in \text{Br}(Y)[2]$. It turns out that $D^b(X, \alpha) \cong D^b(Y, \beta)$. This way we get a good supply of surfaces on which to test Question 2.1. Our goal is then to

(1) Produce a supply of $(X, \alpha)$ and $(Y, \beta)$ as above over $\mathbb{Q}$, in such a way that $X(\mathbb{Q}) = \emptyset$ on account of the class $\alpha$. The delicate point here is to do this in a way that the defining equations of $W$ have small coefficients (this will require an implementation of invariant calculations on 2-adic points of $X$). In order to do this, it’d be nice to guarantee that $\rho(X) = 1$ (this will ensure that $\rho(Y) = 1$, and thus there is no “interference” from algebraic Brauer classes).

(2) For the surfaces in our catalogue, does $\beta$ obstruct rational points on $Y$? If not, can we develop an efficient algorithm to search for points on K3 surfaces of degree 2?

REFERENCES


Department of Mathematics MS 136, Rice University, Houston, TX 77005, USA

E-mail address: varilly@rice.edu

URL: http://www.math.rice.edu/~av15