$L(\mathcal{B}; F, G; \alpha, \beta)$ is continuous

Let F and G be homogeneous coprime square-free polynomials in $k[U, V]$ where k is a number field. $\mathcal{B}$ is the set of bad primes which consists of primes dividing infinity or 2, primes at which any coefficient of F or G is not integral, primes modulo which FG is not separable.

We define

$$L(\mathcal{B}; F, G, \alpha, \beta) = \prod_{p \notin \mathcal{B}, p | G(\alpha, \beta)} (F(\alpha, \beta), G(\alpha, \beta))_p$$

where α, β are integers of k coprime outside $\mathcal{B}$ to each other. Note that $F(\alpha, \beta)$ is unit at each prime occuring in the product.

Define

$$M(\alpha,\beta) = L(\mathcal{B}, F, G, \alpha, \beta)L(\mathcal{B}, U, V, \alpha, \beta)^{(degF)(degG)}.$$

Note that if $deg(F)$ is even, $L = M$.

By the Hilbert product formula,

$$L(f,g)L(g,f) = \prod_{v\in\mathcal{B}} (f(\alpha,\beta), g(\alpha,\beta))_v.$$

So continuity of $L(f,g)$ is same with that of $L(g,f)$.
And if $deg(f) \geq deg(g)$, for any h, a homogeneous polynomial of degree $deg(f) - deg(g)$, $L(f - gh, g) = L(f,g)$.
The same is hold for M.
By the property of Hilbert symbol, $L(fh,g) = L(f,g)L(h,g)$ and $L(f,gh) = L(f,g)L(f,h)$.
So we may assume F and G are irreducible and $deg(F) \geq deg(G)$.

If G is constant, all primes dividing G are already in $\mathcal{B}$. Therefore $L = M = 1$, so continuous.

If G is V, by subtracting every term containing V from F, we may assume $F = aU^r$. But in this case it follow easily that M is continuous.

We will proceed by induction on $deg(F)deg(G)$. By the irreducible assumption, $\delta =$ coefficients of U^r in F and $\gamma =$ the coefficient of U^s in G are non-zero. Let's enlarge $\mathcal{B}$ to $\mathcal{B}_1$ by adjoining the primes dividing the γ. Then,

$$M(\mathcal{B}_1, F, G) = M(\mathcal{B}_1, F - \gamma\delta^{-1}U^{r-s}G, G) = M(\mathcal{B}_1, F', G)M(\mathcal{B}_1, V, G)$$

are continuous in the $\mathcal{B}_1$-adic topology by the induction hypothesis.

The difference of $M(\mathcal{B}_1, F, G)$ and $M(\mathcal{B}, F, G)$ is a product of continuous factors which are continuous in the $\mathcal{B}_1$-adic topology. So $M(\mathcal{B}, F, G)$ is continuous in the $\mathcal{B}_1$-adic topology.

We can find a unimodular linear transformation which make γ', the leading coefficient of G is coprime with γ outside $\mathcal{B}$. If $\mathcal{B}_2$ is the enlargement of $\mathcal{B}$ corresponding to γ', by the same argument $M(\mathcal{B}, F, G)$ is continuous in the $\mathcal{B}_2$-adic topology. But $\mathcal{B}_1 \bigcap \mathcal{B}_2 = \mathcal{B}$.

So $M(\mathcal{B}, F, G)$ is continuous in the $\mathcal{B}$-adic topology. $\square$

A result related to the Schinzel's Hypothesis

Let $P_1(X), \cdots, P_n(X)$ be monic irreducible polynomials over $\mathbb{Q}$ and $\mathcal{B}$ is the set of bad primes consisting of ∞, 2, primes at which a coefficient of $P_i(X)$ is not integral and primes modulo which the reduction of $\prod P_i$ is not separable. Then for given $N \geq \sum deg(P_i)$, there are a number field k with $[k : \mathbb{Q}] = N$ and ξ, an integer in k such that

$$P_i(\xi) = \text{a first degree prime in } k \times \text{junk}_1 \times \text{junk}_2$$

where junk_1 comes from the bad primes and junk_2 has some various possibility. We can choose junk_2 is a square in k to behave nicely on the Hilbert symbol.

The main problem is to find $G(X)$, an irreducible polynomial of degree N with the suitable properties.

Let $R(X) = \prod P_i(X)$, $R_i(X) = \frac{R(X)}{P_i(X)}$

For any polynomial G of degree N, we get

$$\frac{G(X)}{R(X)} = Q(X) + \sum \frac{\psi_i(X)}{P_i(X)}$$

or

$$G(X) = Q(X)R(X) + \sum \psi_i(X)R_i(X).$$

Let λ be a solution of G and λ_i be a solution of P_i. $\phi_i = \psi_i(\lambda_i)$.

Then the resultant of $G(X)$ and $P_i(X)$ is

$\pm \prod(\lambda - \lambda_i) = \pm N_{\mathbb{Q}(\lambda_i)/\mathbb{Q}}(G(\lambda_i)) = \pm N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(P_i(\lambda))$.

But $G(\lambda_i) = \phi_i R_i(\lambda_i)$.

Note that $R_i(\lambda_i)$ is unit outside $\mathcal{B}$. And an approximation argument provides the existence of irreducible polynomial G such that ϕ_i is a product of degree 1 prime outside $\mathcal{B}$ and primes in $\mathcal{B}$ and a squares in $k = \mathbb{Q}(\lambda)$. So the result follows.
Moreover we can choose G arbitrary closed to any given degree N polynomials $G_v(X)$ over $\mathbb{Q}_v$ for all $v \in \mathcal{B}$.

Solubility of Del Pezzo surface of degree 4

A Del Pezzo surface of degree 4, V is realized as an intersection of two quadrics in
P^4. Coray showed that a Del Pezzo surface defined over k of characteristic 0 which contains a point in a extension filed of k with odd extension degree has a rational point.

Lemma If V has a 0-cycle of degree 2 and a 0-cycle of odd degree n, then V is soluble.

Proof We will proceed by the induction on n. Pick integer d such that $2d(d+1) > n > 2d(d-1)$. The hypersufaces in $\mathbb{P}^4$ of degree d give a system of curves on V of dimension $2d(d+1)$.
choose a pencil of curves passing through the original 0-cycle and $\frac{1}{2}(2d(d+1) - n - 1)$ 0-cycles of degree 2.

The self-intersection points of the pencil form a 0-cycle of degree $4d^2$. So we have new 0-cycle of degree $4d^2 - 2d(d+1) - 1 = 2d(d-1) + 1$. So if $d \neq 2d(d-1) + 1$ we have done.
If so, we may choose a pencils of curves passing through the original cycle and $\frac{1}{2}(2d(d+1) - n - 5)$ cycles of degree 2 having a double point at one 2-cycle. Then the 2-cycle is a base point of the pencil with multiplicity 4, and so we get a new 0-cycle of degree $2d(d-1) - 1$. $\square$

The original result comes from the above lemma and the fact that V is a intersection of two quadrics.