Let $V : a_0x_0^2 + a_1x_1^2 + a_2x_2^2 = 0$ be a pencil of conics and c be an irreducible factor of $a_0a_1a_2$.
Define

$$L(\mathcal{B}; -a_0a_1, c; \alpha, \beta) = \prod_{p \notin \mathcal{B}, p | c(\alpha,\beta)} (-a_0a_1, c)_p.$$

$L(\alpha, \beta)$ is continuous in the topology induced by $\mathcal{B}$ on $\mathbb{Z} \times \mathbb{Z}$.
Note that the solubility at (α, β) and the value of $L(\mathcal{B}; -a_0a_1, c)$ depend only on $\lambda = \alpha/\beta$.

If V is soluble at $\alpha \times \beta$, then for each c, $L(\mathcal{B}; -a_0a_1, c; \alpha, \beta) = 1$. The latter condition is equivalent to the Brauer-Manin obstruction.

Theorem If $L(\alpha, \beta) = 1$ for every c thenb there exist a $\alpha' \times \beta'$ near $\alpha \times \beta$ at which V is soluble provided Schinzel's Hypothesis.

Schinzel's Hypothesis

Let $P_1(X), \cdots, P_n(X)$ be polynomials over $\mathbb{Z}$ with positive leading coefficients. Suppose there is a arbitrary large integer n which makes every $P_i(n)$ a prime number, then

- Each P_i is irreducible.
- Set of solutions of $\prod P_i \equiv 0$ modulo p is not all $\mathbb{Z}/p$
  for any P_i and any prime number p.

Schinzel's hypothesis is that the converse is also true.

With the Schinzel's hypothsis, it is easily proved that there exist $\alpha' \times \beta'$ sufficiently closed to $\alpha \times \beta$ in the topology induced by $\mathcal{B}$ such that each $c(\alpha', \beta')$ is a prime up to $\mathcal{B}$.

Proof of Theorem

The solubility on $\mathcal{B}$ and the condition $L(c, \alpha, \beta) = 1$ for each c are open conditions in the topology induced by $\mathcal{B}$. So by the consequence of Schnizel's hypothesis, we can find $\alpha' \times \beta'$ satisfying the solubility on $\mathcal{B}$ and the condition $L = 1$ and each $c(\alpha', \beta')$ is one prime v up to $\mathcal{B}$. Therefore

$$1 = L(c, \alpha', \beta') = (-a_0 a_1, c)_v.$$

The right term means the solubility at $\mathbb{Q}_v$. And there is no problem of solubility at other places. So by the Hasse principle of conics, $V(\alpha', \beta')$ is soluble. $\square$

Let $E' \xrightarrow{\phi} E$ be an isogeny defined over k. The set of coverings C of E which is isomorphic to E' over $\bar{k}$ and make the following diagram commute

$$\begin{tikzcd} E' \arrow[r, "\phi"] \arrow[d, equal] & E \\ C \arrow[ur] & \end{tikzcd}$$

forms a group called the ϕ-Weil-Chatlet group. The subgroup which consist of locally soluble C is ϕ-Selmer group. And the quotient of ϕ-Selmer group over soluble elements is ϕ-Tate-Shafarevich group.

Roughly speaking, ϕ-Tate-Shafrevich group classifies the locally soluble ϕ-covering of E which is not soluble.

Tate-Shafarevich group of E is a kind of union of ϕ-Tate-Shafarevich groups while ϕ varies all isogeny to E.

There is a conjecture that for elliptic curves over number fields, Tate-Shafarevich group is finite.

Caseels showed that there is a non-singular alternating bilinear form on the Tate-Shafarevich group. This gives with the hypothesis that the Tate-shafarevich group of E is finite, if 2-
Selmer group of E is order 8, then every 2-covering of E contains a rational point.

2-Selmer group

Let E be a elliptic curve defined over k whose 2-tortions are all rational given by

$$E : Y^2 = (X - a_1)(X - a_2)(X - a_3).$$

The 2-coverings of E is given by
$\Gamma(m) : m_i Y_i^2 = X - a_i$ for $i = 1, 2, 3$ and $Y = Y_1 Y_2 Y_3$
where $m_1 m_2 m_3$ is a square in k.
The 2-coverings of E is classified by (k^*/k^{*2}) under the correspondence
$\Gamma(m) \mapsto (m_1, m_2)$.
Let $\mathcal{B}$ be the set of bad primes consisting of infinite places, primes dividing 2 and prime dividing $(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$.
The solubility of $\Gamma(m)$ at every good prime not contained in $\mathcal{B}$ is equivalent to that $(m_1, m_2) \in U_{\mathcal{B}} = (o^*_{\mathcal{B}}/o^{*2}_{\mathcal{B}})^2$.

The solubilities at bad primes give a linear conditions and we may have a bilinear form

$$\text{obstructions} \times U_{\mathcal{B}} \to \{\pm 1\}.$$

The Selmer group is realized as the right kernel of the bilinear form.

If we regard a_i and m_i as homogeneous functions with two variables, E is a pencils of elliptic curves and Γ is a pencils of 2-coverings.

With the Schinzel's hypothesis and the hypothesis of Tate-Shafarevich group, there is a criterion of solubility of Γ. Provided suitable conditions on m_i and some condition related to the Brauer-Manin obstruction, for a point of $\mathbb{P}^1$, $\alpha \times \beta$ at which the conics $m_i Y_i^2 - m_j Y_j^2 = (a_j - a_i) Y_0^2$ are soluble, there is $\alpha' \times \beta'$ at which Γ is soluble near $\alpha \times \beta$.

The strategy is enlarging carefully bad primes to reduce the 2-Selmer group of some fiber of Γ until it has order 8.

Some K3 surfaces

Let $a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0$ be a $K3$ surface defined over $\mathbb{Q}$ with the condition $a_0a_1a_2a_3$ is a square.

There is the natural map from the $K3$ surface to the quadric surface $a_0Y_0^2 + a_1Y_1^2 + a_2Y_2^2 + a_3Y_3^2 = 0$. The square condition provide that the two pencils of lines on the quadric is defined over $\mathbb{Q}$ and the pull back of these pencils are pencils of elliptic curves of kinds considered above. So we can apply the argument to the $K3$ surface and we have

Theorem With the Schinzel's hypothesis and the hypothesis of Tate-Shafarevich group, the Brauer-Manin obstruction is the only obstruction of the Hasse principle for the $K3$ surfaces given as the above types.