Grothendieck-Teichmüller theory

- $G_{\mathbb{Q}} = \varprojlim_{\text{certain finite groups } G} G$
- $1 \to Gal(\bar{\mathbb{Q}}/\mathbb{Q}^{ab}) \to G_{\mathbb{Q}} \to Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \to 1$
- $\mathbb{Q}^{ab} = \mathbb{Q}(\text{all roots of unity})$
- $Gal(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^* = \varprojlim (\mathbb{Z}/n\mathbb{Z})^*$
- Shafarevich conjecture $\Rightarrow Gal(\bar{\mathbb{Q}}/\mathbb{Q}^{ab})$ free.

- X is an algebraic variety over $\mathbb{Q}$
- Then if $\bar{\mathbb{Q}}(X)$ denotes the function field of $X/\bar{\mathbb{Q}}$
-

$$
\begin{array}{cccc}
\bigcup_Y \bar{\mathbb{Q}}(Y) = \widetilde{\bar{\mathbb{Q}}(X)} & \tilde{X} & \text{universal cover} \\
\big| \hat{\pi}_1 & \pi_1 \big| & Y \\
\bar{\mathbb{Q}}(X) & X/\bar{\mathbb{Q}} & \\
\big| G_{\mathbb{Q}} & \big| & \\
\mathbb{Q}(X) & X/\mathbb{Q} &
\end{array}
$$

- $\hat{\pi_1} = Gal(\widetilde{\bar{\mathbb{Q}}(X)}/\bar{\mathbb{Q}}(X) = \varprojlim_{\substack{N \trianglelefteq \pi_1 \\ \pi_1/N \text{ finite}}} \pi_1/N$
- $1 \to \hat{\pi_1} \to Gal(\widetilde{\bar{\mathbb{Q}}(X)}/\bar{\mathbb{Q}}(X)) \to G_{\mathbb{Q}} \to 1$
- $G_{\mathbb{Q}} \to Out(\hat{\pi_1})$ canonical

Grothendieck's Vision

- get information on $G_{\mathbb{Q}}$ by using the group structure of the π_1's.
- 1) $\mathbb{P}^1 - \{0, 1, \infty\}$
- 2) Genus zero moduli spaces
- 3) Moduli spaces
- $\chi : G_{\mathbb{Q}} \to \hat{\mathbb{Z}}^* = \varprojlim (\mathbb{Z}/n\mathbb{Z})^*$ defined by $\sigma(\zeta_n) = \zeta_n^{\chi(\sigma)}$

Fact The $G_{\mathbb{Q}}$ action on a fundamental group preserves inertia up to conjugacy.

- Inertia subgroups are cyclic: $\langle\alpha\rangle \subset \hat{\pi_1}$
- $\sigma(\alpha) = \gamma^{-1}\alpha\gamma$
- $\sigma(\langle\alpha\rangle) = \gamma^{-1}\langle\alpha\rangle\gamma$

Example $\mathbb{P}^1 \setminus \{0,1,\infty\}$

[Drawing of points 0,1, and ∞, with loops around them labeled x,y, and z. The loops touch the tip of a vector based at the 0 point labeled $\overrightarrow{01}$.]

- $xyz = 1$
- Inertia in $\pi_1(X, \overrightarrow{01})$ is $\langle x \rangle$, $\langle y \rangle$, $\langle z \rangle$

- $G_{\mathbb{Q}} \to Out(\hat{F}_2)$
- $\sigma \in G_{\mathbb{Q}}$. Choose a lifting $\sigma \in Aut(\hat{F}_2)$ s.t.
  - $\sigma(x) = x^{\alpha}$
  - $\sigma(y) = \tilde{f}^{-1} y^{\beta} \tilde{f}$
  - $\sigma(z) = \tilde{g}^{-1} z^{\gamma} \tilde{g}$

- By abelianizing, we can see that $\alpha = \beta = \gamma$, so $\sigma(x) = x^\alpha$, $\sigma(y) = \tilde{f}^{-1} y^\alpha \tilde{f}$.
- Modifying my lift by $Inn(X^*)$, i.e. modifying the choice of $y^n \tilde{f} x^m$ ...

- ... gives another lift of σ. But in $\{y^m \tilde{f} x^n | m, n \text{ of } X\}$, $\exists$ unique f which is in $\hat{F}_2'$.
- $\hat{F}_2'$ = derived subgroup of $\hat{F}_2$.
- $f \in ker(\hat{F}_2 \to mat\hat{h}bbZ)$. $x \mapsto 0$, $y \mapsto 1$.
- $f \in ker(\hat{F}_2 \to mat\hat{h}bbZ)$. $x \mapsto 1$, $y \mapsto 0$.

- So to every $\sigma \in G_{\mathbb{Q}}$, I have associated a pair $(\alpha, f) \in \hat{\mathbb{Z}} \times \hat{F}_2'$
- $\alpha = \chi(\sigma)$

- So $G_{\mathbb{Q}} \to \hat{\mathbb{Z}} \times \hat{F}_2'$.
- $Gal(\hat{\mathbb{Q}}/\mathbb{Q}^{ab}) \to \hat{F}_2'$
- Not a group hom with usual group law in $\hat{\mathbb{Z}}^\times \times \hat{F}_2'$
- New group law: $(\lambda, f) \mapsto x^\lambda, f^{-1}y^\lambda f$ endomorphism of $\hat{F}_2$.
- $F := f^{-1}y^\lambda f$. Compose endomorphisms.

- $(\lambda, f)(\mu, g) = (\lambda\mu, fF(g))$

Interlude: Dessins d'enfants

- A combinatorial way of looking at the Galois action of $\hat{F}_2 = \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{01})$
- **Def** A Dessin is a finite collection of vertices and edges on a topological surface of genus g.
  - Connected

- – It cuts the surface into cells.
  – You can color the vertices in 2 colors in such a way that all neighbors of a vertex have the opposite color.

[some children's drawings such as a house, a stick figure, a subdivided torus]

- Dessins are in bijection with conjugacy classes of finite index subgroups of F_2, which are in bijection with finite covers of $\mathbb{P}^1 - \{0, 1, \infty\}$.

- The open finite index subgroups of $\hat{F}_2$ (under conjugacy) $\leftrightarrow$ covers $\leftrightarrow$ dessins.
- But $G_{\mathbb{Q}}$ acts on the former, so it acts on dessins.

- Goal of Dessin theory: combinatorially determine the Galois orbit of a dessin, i.e. give combinatorial invariants of dessins which describe the Galois orbit.

$$\begin{array}{ccl} \beta^{-1}([0,1]) & X & \text{ramified only over } 0,1,\infty \\ & |\beta & \\ [0,1] & \mathbb{P}^1 & \end{array}$$

[Drawing of a stick figure with black and white vertices]

- Valencies:
  - $\bullet(3,1,1,3)$
  - $\circ(1,1,4,2)$
  - $\infty(2,6)$
- So the set of dessins which might be in the Galois orbit is finite, since they must have the same valency lists.

Invariants

- Galois group of dessin: Galois group of the Galois closure of the cover $\begin{pmatrix} cX \\ | \\ \mathbb{P}^1 \setminus \{0,1,\infty\} \end{pmatrix}$
- Build up dessins by composing - creates new Galois invariants: $\begin{pmatrix} cX \\ | \\ \mathbb{P}^1 \\ | \\ \mathbb{P}^1 \end{pmatrix}$
- [two trees, each with a central vertex of valence 5, and its adjacent vertices of valences 2, 3, 4, 5, and 6, but the valence 2 and 3 vertices are in opposite orders.]
- $(n, 1, \ldots, 1)$

- $(a_1, \ldots, a_n)$
- Kotchakov: If $(a_1 + \cdots + a_n)a_1 \ldots a_n =$ square, then the dessins "diameter 4 trees" break up into 2 Galois orbits: even permutations and odd permutations of the branches.
- proved by Zappon.