Prop $$S_k(x,f) = \sum_{x_i^{q^k-1}=1} F_a(f,x)\, F_a(f,x^q) \cdots F_a(f,x^{q^{k-1}}).$$

5. p-adic Banach space.

Write $f = \sum_{j=1}^{J} a_j x^{V_j}$, $a_j \in \mathbb{Z}_q$, $a_j^{q-1} = 1$.

$\Delta = \Delta(f) = \mathrm{Conv}(V_j)$. in $\mathbb{R}^n$

$\bar{\Delta} = \mathrm{Conv}(0, (1, V_j))$ in $\mathbb{R}^{n+1}$.

$C(\bar{\Delta}) =$ Cone gen'ed by $\bar{\Delta}$.

$L(\bar{\Delta}) = C(\bar{\Delta}) \cap \mathbb{Z}^{n+1}$. a f.g. monoid.

Def. $S_{\Delta,q} = \left\{ \sum_{u \in L(\tilde{\Delta})} A_u \pi^{u_0} X^u \;\middle|\; A_u \in \mathbb{Z}_q[\pi] \right\}$. a p-adic Banach alg.

$u = (u_0, u_1, \ldots u_n)$, $\quad X^u = X_0^{u_0} X_1^{u_1} \cdots X_n^{u_n}$.

Formal basic $\quad \Gamma = \{ \pi^{u_0} X^u \mid u \in L(\tilde{\Delta}) \}$.

Def. $\phi_1 = \psi_p \circ F(f, x)$. τ^{-1}- linear.

$\phi_a = \psi_q \circ F_a(f, x) = \phi_1^a$, linear,

$\Rightarrow$ ϕ_a is a compact operator on $S_{\Delta, q}$.

$Tr(\phi_a)$ is defined

$\det(I - T\phi_a)$ is p-adic entire.

$$\phi_1(\Gamma) = \Gamma A_1(f).$$

$$A_1(f) = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ p\,a_{10} & p\,a_{11} & p\,a_{12} & \cdots \\ p^2 a_{20} & p^2 a_{21} & p^2 a_{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

a_{ij}.

$\ell(i\Delta)$ – rows

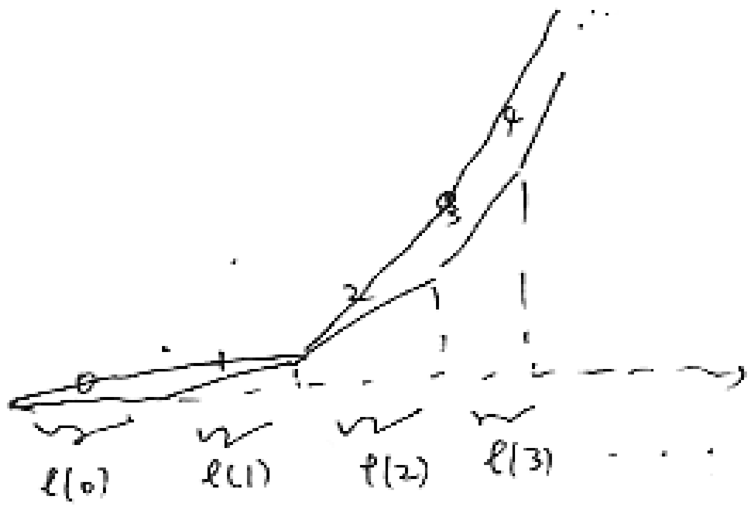
$\ell(j\Delta)$ – columns

entries in $\mathbb{Z}_q[\pi]$.

<u>Prop</u>. q-adic NP of $\det(I - \phi_a T)$

$\geq$ the polygon in $\mathbb{R}^2$ with vertices

$$\left(\sum_{k=0}^{m} \ell(k), \sum_{k=0}^{m} k\,\ell(k) \right); \quad m = 0, 1, 2, \ldots$$



$$\ell(i) = \#(i\Delta \cap \mathbb{Z}^n)$$

7. Dwork trace formula.

Thm $\quad S_k(x_0 f) = (q^k - 1)^{n+1} \, \mathrm{Tr}(\phi_a^k)$

$$= \sum_{i=0}^{n+1} (-1)^{n+1-i} \binom{n+1}{i} q^{ki} \, \mathrm{Tr}(\phi_a^k).$$

$$\Rightarrow L(x_0 f, T) = \prod_{i=0}^{n+1} \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \mathrm{Tr}\left((\delta^i \phi_a)^k \right) \right)^{(-1)^{n+1-i}\binom{n+1}{i}}$$

$$= \prod_{i=0}^{n+1} \det\left(I - T \delta^i \phi_a \right)^{(-1)^{n-i}\binom{n+1}{i}}$$

Cor. $L(x_0 f, T)$ is p-adic mero. in T.

$\Uparrow$

$1 + T\mathbb{Z}[[T]]$

8. Rationality.

Lemma (Borel – Dwork). Let $g(T) \in \mathbb{Z}[[T]]$.

Then $g(T) \in \mathbb{Q}(T) \Longleftrightarrow$

$$\begin{cases} 1) & g(T) \text{ analytic } \text{~~near~~ at } 0 \text{ ~~at~~ in } \mathbb{C}. \\ 2) & g(T) \text{ is } p\text{-adic mero for some } p > 0 \end{cases}$$

$\Rightarrow$

<u>Thm</u> (Dwork). $L(\chi\circ f, T) \in \mathbb{Q}(T)$.

$Z(U_f, T) \in \mathbb{Q}(T)$

9. p-adic coh formula for $L(\chi_0 f, T)$.

<u>Def</u>. $$G(x) = F(f, x)\, F^{\tau}(f, x^{p})\, F^{\tau^2}(f, x^{p^2}) \cdots \quad \in \mathbb{Z}_q[\pi][[X_0, X_1^{\pm 1}, \cdots]]$$

$$F(f, x) = \frac{G(x)}{G^{\tau}(x^{p})}.$$

$$\psi_p(x^{u}) = \begin{cases} x^{\frac{u}{p}}, & \text{if } p \mid u \\ 0, & \text{if } p \nmid u. \end{cases}$$

$$\phi_1 = \psi_p \circ F(f,x) = \psi_p \circ \frac{G(x)}{G^{\tau}(x^p)} = G(x)^{-1} \circ \psi_p \circ G(x)$$

$$\phi_1^a = \phi_a = G(x)^{-1} \cdot \psi_p^a \circ G(x) = G(x)^{-1} \circ \psi_q \circ G(x).$$

For $0 \le i \le n$. let $D_i = G(x)^{-1} \circ x_i \frac{\partial}{\partial x_i} \cdot G(x)$ acts on $S_{\Delta, q}$.

$$D_i D_j = D_j D_i, \qquad \phi_a \circ D_i = q D_i \circ \phi_a$$

Koszul complex $K_\bullet(S_{\Delta,q}, D_0, D_1, \ldots D_n)$.

$$\begin{array}{ccccccccccc}
0 \to & S_{\Delta,q}^{\binom{n+1}{n+1}} & \xrightarrow{d} & S_{\Delta,q}^{\binom{n+1}{n}} & \xrightarrow{d} & \cdots & \xrightarrow{d} & S_{\Delta,q}^{\binom{n+1}{1}} & \xrightarrow{d} & S_{\Delta,q} & \to 0 \\
& \downarrow q^{n+1}\cdot\phi_a & & \downarrow q^n\cdot\phi_a & & & & \downarrow q\cdot\phi_a & & \downarrow \phi_a & \\
0 \to & S_{\Delta,q}^{\binom{n+1}{n+1}} & \to & S_{\Delta,q}^{\binom{n+1}{n}} & \to & \cdots & & S_{\Delta,q}^{\binom{n+1}{1}} & \to & S_{\Delta,q} & \to 0
\end{array}$$

$$\Rightarrow L(X_0 f, T)^{(-1)^n} = \prod_{i=0}^{n+1} \det\left(I - Tq^i\phi_a \mid S_{\Delta,q}\right)^{(-1)^i\binom{n+1}{i}}$$

$$= \prod_{i=0}^{n+1} \det\left(I - Tq^i\phi_a \mid H_i(K_\bullet)\right)^{(-1)^i}.$$

If f is Δ-regular $/\mathbb{F}_q$. $\Longrightarrow$ $\begin{cases} H_i = 0 & \forall\ i > 0 \\ H_0 = S_{\Delta,q} / \sum_{i=0}^{n+1} D_i(S_{\Delta,q}). \end{cases}$

is a free $\mathbb{Z}_q[\pi]$-module of rk $d(\Delta)$

From now on, assume f is Δ-regular. $\Rightarrow$

<u>Thm</u> $L(\chi_0 f, T)^{(-1)^n} = \det(I - T\phi_a \mid H_0)$

$\in 1 + T\mathbb{Z}[T]$ of deg $d(\Delta)$

10). $$L(x_0 f, T)^{(-1)^n} = \prod_{i=1}^{d(\Delta)} (1-\alpha_i T), \quad \alpha_i \in \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

$$\Rightarrow \quad |\alpha_i| = q^{\frac{w_i}{2}}, \quad w_i \in \mathbb{Z} \cap [0, n+1].$$

(mixed of weight $\le n+1$).

<u>Def</u>. $$e_j = \#\{1 \le i \le d(\Delta) \mid w_i = j\}, \quad j = 0, 1, \dots, n+1$$

The weight polygon of Δ is $(e_0 = 1)$.

$WP(\Delta)$:

$wp(\Delta)$ can be determined.

Eg Δ a simplex, let

$$c_0 = 1, \qquad c_i = \sum_{\substack{\sigma \subset \Delta \\ \dim \sigma = i-1}} \mathrm{Vol}(\sigma), \quad i \geq 1$$

$$\Rightarrow e_0 = 1, \qquad e_j = \sum_{i=0}^{j} (-1)^{j-i}\, i! \binom{n+1-i}{n+1-j} c_i, \quad j \geq 1.$$

Ex. $f(\lambda, x) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda.$ Δ - regular.

II. Newton polygon.

$$L(x_0 f, T)^{(-1)^n} = \prod_{i=1}^{d(\Delta)} (1-\alpha_i T), \quad \alpha_i \in \overline{Q_p}$$

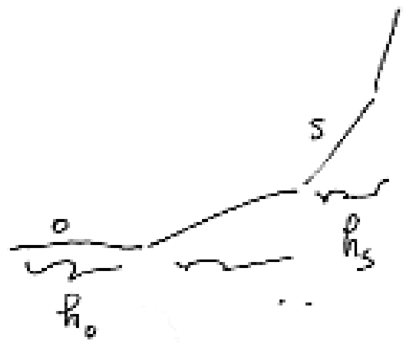
$$|\alpha_i|_q = q^{-s_i}, \quad s_i = \mathrm{ord}_q(\alpha_i).$$

$$s_i \in Q \cap [0, n+1].$$

<u>Def.</u> $h_s = \# \{ 1 \le i \le d(\Delta) \mid s_i = s \}$, $s \in Q \cap [0, n+1]$

q-adic NP.

NP(f):



$\Sigma h_s = d(\Delta)$.

<u>Question</u>: NP(f) = ?

Prop. write

$$L(x,f,T)^{(-1)^n} = \sum_{m=0}^{d(\Delta)} A_m T^m, \quad A_m \in \mathbb{Z}.$$

$\Rightarrow NP(f)$ is convex closure in $\mathbb{R}^2$ of

the pts $(m, \mathrm{ord}_q(A_m))$, $m = 0, 1, \ldots, d(\Delta)$

Prop. Vertices of $NP(f) \in \mathbb{Z}^2$.

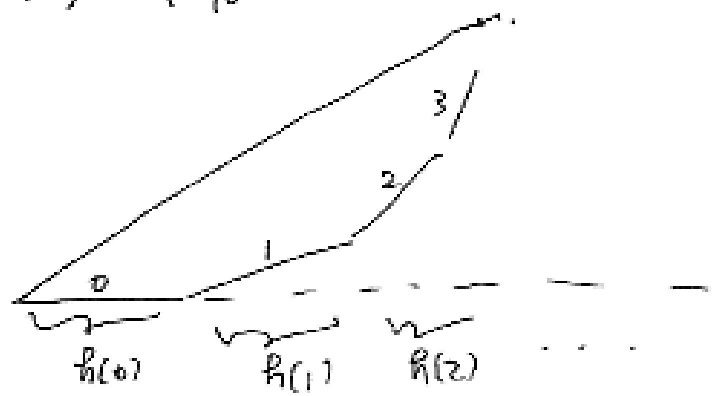
(2). Hodge polygon.

$\Delta \subset \mathbb{R}^n$, n-dim integral convex

$$W(k) = \#(\mathbb{Z}^n \cap k\Delta).$$

$$\sum_{k=0}^{\infty} W(k) T^k = \frac{\sum_{k=0}^{n} h(k) T^k}{(1-T)^{n+1}}.$$

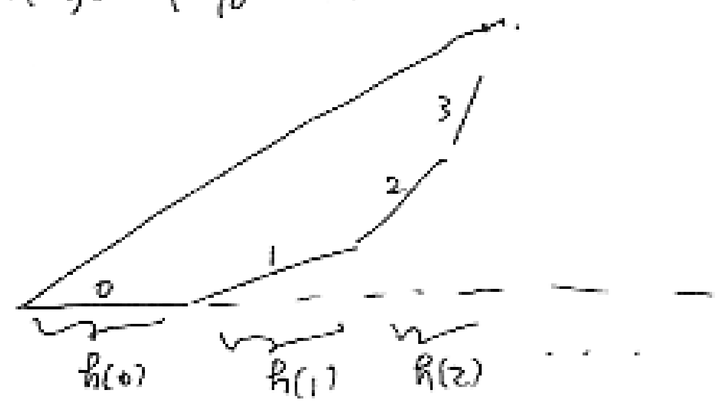
Def. The Hodge polygon of Δ is the polygon



$$\sum h(k) = d(\Delta)$$

Thm. $NP(f) \geq HP(\Delta)$ with endpts coincide.

Def. The Hodge polygon of Δ is the polygon.



$$\sum h(k) = d(\Delta)$$

Thm. $NP(f) \geq HP(\Delta)$ with endpts coincide.

13). Variation of $NP(f)$ with p.

Conj. Let $f(x_1, \cdots, x_n) \in \mathbb{Q}[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$, Δ-regular.

1) $\exists$ infinitely many p s.t. $NP(f \otimes \mathbb{F}_p) = HP(\Delta)$.

2) $$\delta(f) = \lim_{t \to \infty} \frac{\#\{p \le t \mid f \otimes \mathbb{F}_p \text{ is ordinary}\}}{\#\{p \le t\}}$$

exists and $\delta(f) > 0$.

$\underline{\text{Eg}}$ $f = X_1 + X_2 + \frac{1}{X_1 X_2} \rightsquigarrow \in \mathbb{Q}[X_1^{\pm 1}, X_2^{\pm 1}]$, Δ-regular.

(elliptic curve $/\mathbb{Q}$).

$$\Rightarrow \delta(f) = \begin{cases} \frac{1}{2}, & f \text{ has CM} \quad (\text{Deuring}) \\ 1, & f \text{ has non-CM} \quad (\text{Serre}) \end{cases}$$

Eg. $f(\lambda, x) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n} - \lambda$, Δ-regular.

$n \geq 4$.

$n = 3$ (surface, K_3)