1. Zeta Functions

$\mathbb{F}_q$, $q = p^a$, p prime

$f(x_1, \ldots x_n) \in \mathbb{F}_q[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$U_f = \{(x_1, \ldots x_n) \in \mathbb{G}_m^n \mid f(x) = 0\}$.

$U_f(\mathbb{F}_q) = \{(x_1, \ldots x_n) \in \mathbb{F}_q^{*n} \mid f(x) = 0\}$.

$U_f(\mathbb{F}_{q^k})$, $k = 1, 2, 3, \ldots$

Def. $Z(U_f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} \# U_f(\mathbb{F}_{q^k})\right)$

$\in 1 + T\mathbb{Z}[[T]]$

Basic Properties/Questions.

1) $Z(V_f, T) \in \mathbb{Q}(T)$.

$$Z(V_f, T) = \frac{\prod_{i=1}^{d_1}(1-\alpha_i T)}{\prod_{j=1}^{d_2}(1-\beta_j T)}, \quad \alpha_i, \beta_j \in \overline{\mathbb{Q}}$$

$$\# V_f(\mathbb{F}_{q^k}) = \sum \beta_j^k - \sum \alpha_i^k,$$

$k = 1, 2, 3 \ldots$

$$\Rightarrow \#V_f(\mathbb{F}_{q^k}) = \frac{(q^k-1)^n+(-1)^{n+1}}{q^k} + (-1)^{n+1}(\beta_0^k+\beta_1^k+\cdots+\beta_{d(\Delta)-2}^k).$$

$$\Rightarrow \quad |\beta_j| \le q^{\frac{n-1}{2}}, \qquad |\beta_j| = \sqrt{q}^{\,w_j}, \qquad w_j \in [0, n-1]\cap\mathbb{Z}.$$

(weight)

w_j can be determined.

$|\beta_j|_\ell = 1$. $\quad \ell$ prime $\neq p$

$\mathrm{ord}_\ell \beta_j = 0$.

3). f Δ-regular, $\Rightarrow$

$|\beta_j|_q = q^{-s_j}$. $\quad s_j \in [0, n-1]\cap\mathbb{Q}$

$s_j = ?$ (q-adic slope of β_j).

(related to Hodge numbers).

4). How s_j and $Z(V_f, T)$ vary as f varies?

5) What more can be said about $Z(V_f, T)$ if f is a CY.

2. L-function of exp sums.

Let $\psi: \quad \mathbb{F}_p \longrightarrow \mathbb{C}^*$

$$x \longrightarrow \exp\left(\frac{2\pi i x}{p}\right)$$

$$\mathbb{F}_{q^k} \xrightarrow{\mathrm{Tr}} \mathbb{F}_p \xrightarrow{\psi} \mathbb{C}^*$$

$\psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}$:

$$S_k(x_0 f) = \sum_{\substack{x_i \in \mathbb{F}_{q^k}^* \\ 0 \le i \le n}} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p} (x_0 f) \in \mathbb{Z}$$

$$q^k \# U_f(\mathbb{F}_{q^k}) = \sum_{\substack{x_i \in \mathbb{F}_{q^k}^* \\ 1 \le i \le n}} \sum_{x_0 \in \mathbb{F}_{q^k}} \psi \circ \mathrm{Tr}(x_0 f)$$

$$= (q^k - 1)^n + S_k(x_0 f)$$

Def. $L(x_0 f, T) = \exp\left(\sum_{k=1}^{\infty} \frac{T^k}{k} S_k(x_0 f) \right)$

$\Rightarrow \quad Z(U_f, 8T) = Z(G_m^n, T) \cdot L(x_0 f, T)$

$\Rightarrow$ enough to study $L(x_0 f, T)$.

3. Dwork's p-adic character.

<u>Def</u>. The Artin – Hasse series

$$E_p(T) = \exp\left(T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots\right)$$

$$= \prod_{(k,p)=1} (1 - T^k)^{-\frac{\mu(k)}{k}}, \quad \mu = \text{Möbius}.$$

$$\in 1 + T(\mathbb{Z}_p \cap \mathbb{Q})[[T]].$$

$\Rightarrow$ $E_p(T)$ converges in $|T|_p < 1$. NOT on $|T|_p \leq 1$.

Def. Let π be a fixed root of

$$T + \frac{T^p}{p} + \frac{T^{p^2}}{p^2} + \cdots = 0 \quad \text{in } \overline{\mathbb{Q}_p}$$

s.t. $\operatorname{ord}_p(\pi) = \frac{1}{p-1}$ (exactly $p-1$ such roots)

$\mathbb{Q}_p(\pi) = \mathbb{Q}_p(\zeta_p)$, π is a uniformizer. $\pi \sim 1 - \zeta_p$

Def. $\theta(T) = E_p(\pi T)$ is convergent in $|T|_p < p^{\frac{1}{p-1}}$.

$$= 1 + \pi T + \cdots$$

Prop. $\theta(1) = 1 + \pi \pmod{\pi^2}$ $\quad \theta(1) \neq 1.$

$\theta(1)^p = 1.$

Def. $\psi : \bar{\mathbb{F}}_p \longrightarrow \mathbb{C}_p^*$

$\bar{x} \rightarrow \theta(1)^{\bar{x}} = \theta(x), \quad x = \mathrm{Teich}(\bar{x}). \quad x^p = x.$

$\psi \circ \mathrm{Tr}_{\mathbb{F}_{p^k}/\mathbb{F}} : \mathbb{F}_{p^k} \xrightarrow{\mathrm{Tr}} \mathbb{F}_p \xrightarrow{\psi} \mathbb{C}_p^*. \quad \psi \circ \mathrm{Tr}_{\mathbb{F}_{p^k}/\mathbb{F}_p}(\bar{x}) = \theta(x)\theta(x^p)\cdots\theta(x^{p^{k-1}}).$

4. p-adic rep of $S_k(x_0 f)$.

Write $x_0 \bar{f} = \sum_{j=1}^{J} \bar{a}_j x_0 x^{V_j}$, $\bar{a}_j \in \mathbb{F}_q$.

$\in \mathbb{F}_q[x_0, x_1^{\pm 1}, \cdots x_n^{\pm 1}]$. $q = p^a$.

$$S_k(x_0 f) = \sum_{x_i \in \mathbb{F}_{q^k}^*} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(x_0 f).$$

$$= \sum_{x_i \in \mathbb{F}_{q^k}^*} \prod_{j=1}^{J} \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}\left(\tilde{a}_j x_0 x^{V_j}\right)$$

$$= \sum_{x_i^{q^k-1}=1} \prod_{j=1}^{J} \prod_{i=0}^{ak-1} \theta\left((a_j x_0 x^{V_j})^{p^i}\right), \quad q^k = p^{ak}$$

$$S_k(x_0 f) = \sum_{x_i \in \mathbb{F}_{q^k}^*} \psi_0 \operatorname{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(x_0 \bar{f}).$$

$$= \sum_{x_i \in \mathbb{F}_{q^k}^*} \prod_{j=1}^{J} \psi_0 \operatorname{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}\left(\tilde{a}_j x_0 x^{V_j}\right)$$

$$= \sum_{x_i^{q^k-1}=1} \prod_{j=1}^{J} \prod_{i=0}^{ak-1} \theta\left((a_j x_0 x^{V_j})^{p^i}\right), \quad q^k = p^{ak}$$