Periods

$$\mathcal{M} : P = \sum_{i=1}^{5} x_i - 5\psi \prod_{i=1}^{5} x_i = 0 \quad \text{in} \;\; \mathbb{P}^4$$

is Calabi-Yau:

$$c_i = 0 \Leftrightarrow \exists \;\text{ a holomorphic } (3,0) - \text{form } \;\Omega$$

$$x_i \to \alpha^{n_i} x_i, \qquad \alpha^5 = 1, \qquad \sum n_i = 0 \text{ mod } 5$$

$$G \cong \mathbb{Z}_5^3$$

$$P \to P + \sum c_{\mathbf{v}} x^{\mathbf{v}}, \qquad x^{\mathbf{v}} = \prod x_i^{v_i}, \qquad \sum v_i = 5, \qquad \mathbf{v} \neq (1,1,1,1,1)$$

$\mathbf{v}$		
$(1,1,1,1,1)$	1	
$(4,1,0,0,0)$	20	
$(3,2,0,0,0)$	20	
$(3,1,1,0,0)$	30	
$(2,2,1,0,0)$	30	
$(2,1,1,1,0)$	20	interior to codim 1 faces of Δ
$(5,0,0,0,0)$	5	vertices

$$x^i \to x^i + \xi^i(x), \quad P(x) \to P(x+\xi) = P(x) + \xi^i P_i(x), \quad \mathcal{I} = (P_i(x))$$

$$h^{2,1} = 101$$

$$h^{p,q} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$$\Omega = \frac{1}{2\pi i} \oint \frac{\epsilon_{ijklm} x^i dx^j dx^k dx^l dx^m}{P(x)} = \frac{\epsilon_{ijklm} x^i dx^j dx^k dx^l}{\partial P(x)/\partial x^m}$$

Canonical set of coordinates: $\int_{T_j} \Omega$

Example:

$$f_0 = \frac{-5\psi}{(2\pi i)^3} \int_{C_2 \times C_3 \times C_4} \frac{x^1 dx^2 dx^3 dx^4}{\partial P/\partial x^5}$$

where

$$C_i = \{|x_i| = \epsilon\}$$

$$f_0 = \frac{-5\psi}{(2\pi i)^4} \int\limits_{C_2\times\cdots\times C_5} \frac{x^1 dx^2 \cdots dx^5}{P} = \frac{-5\psi}{(2\pi i)^5} \int\limits_{C_1\times\cdots\times C_5} \frac{dx^1 \cdots dx^5}{P} =$$

$$= \frac{-5\psi}{(2\pi i)^5} \int\limits_{C_1\times\cdots\times C_5} \frac{dx^1 \cdots dx^5}{-(5\psi \prod x_i)(1 - \frac{\sum x_i^5}{5\psi \prod x_i})} =$$

$$= \frac{1}{(2\pi i)^5} \int\limits_{C_1\times\cdots\times C_5} \frac{dx^1 \cdots dx^5}{\prod x_i} \sum_{r=0}^{\infty} \frac{(\sum x_i^5)^r}{(5\psi \prod x_i)^r} = \sum_{m=0}^{\infty} \frac{(5m)!}{m!^5} \lambda^m; \;\; \lambda = \frac{1}{(5\psi)^5}$$

If $\theta = \lambda \frac{d}{d\lambda}$, then $\mathcal{L} f_0 = 0$ for $\mathcal{L} = \theta^4 - 5\lambda \prod_{i=1}^{4}(5\theta + i)$.

Periods can also be realized as

$$\int_\Gamma d^5x \frac{x^{\mathbf{m}}}{P^{1+\frac{1}{5}\deg \mathbf{m}}}$$

where $\Gamma = C_1 \times \cdots \times C_5$, C_i a contour around $\{P_i = 0\}$, $\mathbf{m}$ any monomial of degree divisible by 5.

The $x^{\mathbf{v}}$, $Qx^{\mathbf{v}}$ where $Q = x_1x_2x_3x_4x_5$ transform in a certain way under G.

$1 \to Q \to Q^2 \to Q^3$ are associated to the 4 solutions of $\mathcal{L}$.

$x^{\mathbf{v}} \to Qv^{\mathbf{v}}$ give 100 differential equations $\mathcal{L}_{\mathbf{v}}$ of degree 2.

$$\frac{1}{5}\frac{d}{d\psi}\frac{1}{P} = \frac{Q}{P^2}$$

It suffices to consider

$$\int d^5x \left\{ \frac{1}{P}, \frac{Q}{P^2}, \frac{Q^2}{P^3}, \frac{Q^3}{P^4} \right\}$$

and

$$\int d^5x \left\{ \frac{x^{\mathbf{v}}}{P^2}, \frac{Qx^{\mathbf{v}}}{P^3} \right\}$$

These satisfy

$$\mathcal{L}_{\mathbf{v}} = \theta(\theta - 1 + c_{\mathbf{v}}) - 5^5\lambda(\theta + a_{\mathbf{v}})(\theta + b_{\mathbf{v}})$$

which is

$${}_2F_1(a_{\mathbf{v}}, b_{\mathbf{v}}, c_{\mathbf{v}}, \psi^{-5})$$

$\mathbf{v}$	$(a_{\mathbf{v}}, b_{\mathbf{v}}, c_{\mathbf{v}})$
$(4,1,0,0,0)$	$(2/5, 3/5, 1)$
$(3,2,0,0,0)$	$(1/5, 4/5, 1)$
$(3,1,1,0,0)$	$(1/5, 3/5, 4/5)$
$(2,2,1,0,0)$	$(1/5, 2/5, 3/5)$

We want to solve $\mathcal{L}\varpi = 0$, we use the method of Frobenius

$$\varpi(\lambda, \epsilon) = \sum_{n=0}^{\infty} A_n(\epsilon)\lambda^{n+\epsilon}$$

Since

$$\theta\lambda^m = m\lambda^m$$

we get

$$(n+\epsilon)^4 A_n(\epsilon) - 5\prod_i (5(n+\epsilon) - 5 + i) A_{n-1}(\epsilon) = 0$$

$$A_n(\epsilon) = \frac{5(n+\epsilon)\prod_{i=1}^{4}(5(n+\epsilon) + i - 5)}{(n+\epsilon)^5} A_{n-1}(\epsilon)$$

$$A_n(\epsilon) = \frac{\Gamma(5n+5\epsilon+1)}{\Gamma^5(n+\epsilon+1)}\frac{\Gamma^5(\epsilon+1)}{\Gamma(5\epsilon+1)}$$

ϖ and its first 3 derivatives solve the differential equation.

$$\varpi_0 = f_0$$

$$\varpi_1 = f_0 \log \lambda + f_1$$

$$\varpi_2 = f_0 \log^2 \lambda + 2 f_1 \log \lambda + f_2$$

$$\varpi_3 = f_0 \log^3 \lambda + 3 f_1 \log^2 \lambda + 3 f_2 \log \lambda + f_3$$

Problem:

Consider $P(x) = 0$, $x \in \mathbb{F}_p^5$, $\psi \in \mathbb{F}_p$, $p \neq 5$.

Let $\nu(\lambda) = \#\{P(x) = 0 | x \in \mathbb{F}_p^5\}$, $\lambda = \frac{1}{(5\psi)^5}$. How does it vary with λ?

Write $\nu = \nu_0 + \nu_1 p + \nu_2 p^2 + \nu_3 p^3 + \nu_4 p^4$, $0 \leq \nu_i \leq p - 1$.

$$\nu = \sum_{x \in \mathbb{F}_p^5} (1 - P(x)^{p-1}) = \sum_{m=0}^{[p/5]} \frac{(5m)!}{m!^5} \lambda^m$$