Toric Mirror Symmetry

- X and Y Calabi-Yau varieties.
- The corresponding conformal field theories are sometimes “isomorphic.”
- Similar example from elementary geometry:
  A lattice cone $\langle e_1, e_2 \rangle$ in a plane, and “half-twist” it
  by shifting the origin to the center of the parallelogram.
- This transformation is in $GL(2, \mathbb{Z})$.
- When can 2 non-isomorphic cones give isomorphic parallelograms?

Cone Duality

- View a parallelogrm as a gluing of two triangles.
- Standard cone generaed by $(1, 0)$ and $(1, n)$.
  Here, all lattice points lie on the drawn diagonal.
- Construct the dual cone of σ by
  $\hat{\sigma} = \{y : \langle x, y\rangle \geq 0 \forall x \in \sigma\}$
- Geometrically, this is done by taking orthogonal vectors to the defining ones for σ.
- For $n \neq 1, 2$ there are non-isomorphic cones, with isomorphic parallelograms.

Toric Mirror Symmetry

- Also based on duality between cones.
- Construct a cone over our given polytope, where the section is defined by a scalar product restriction: $\langle x, u \rangle = 1$.
- We want the property that the dual cone is also over some polytope.
- A *reflexive polytope* is a polytope such that the dual cone of the cone over the polytope can also be viewed as the cone over some other polytope.
- For $d = 1$, we just have intervals. For higher dimensions, there are computer classifications.

Cohomological Interpretation

- A family of polynomials $f(x) = \sum_{m \in A} a_m x^m$, defining a collection of hypersurfaces $Z_f \subset (\mathbb{C}^*)^d$.
- Consider the primitive cohomology: $\dim PH^{d-1}(Z_f) = \mathrm{Vol}(\Delta) - 1 = d(\Delta) - 1$.
- Consider the Hodge filtration: $S_\Delta \to P_\Delta(t) = \sum_{k \geq 0} \ell(k\Delta) = \frac{\psi_0(\Delta)+\psi_1(\Delta)+\cdots+\psi_d(\Delta)t^d}{(1-t)^{d+1}}$
- Here, $\psi_0(\Delta) = 1$ always, and $\psi_i = \dim S_f^i$, recalling that by definition, $S_f = S_\Delta / \langle F_0, \ldots, F_d \rangle$.

- Similarly, we consider $I_\Delta \to Q_\Delta(t) = \sum l^*(k\Delta)t^k$.
- $Q_\Delta = \frac{\phi_\Delta(t)}{(1-t)^{d+1}}$ counts the analogous dimension for $I_f = I_\Delta / \langle F_0, \ldots, F_d \rangle$.
- We also have the pairing $S_f^i \times I_f^{d+1-i} \to I_f^{d+1} \approx \mathbb{C}$.

- Take the polytope $\Delta \subset M_{\mathbb{R}}$ to $(1, \Delta) \subset \tilde{M}_{\mathbb{R}}$, where $\tilde{M}$ is a lattice of rank $d+1$.
- We wish to associate some variety.
- Assume $|A| = n$, and $\{(i, v_i)\} \in \tilde{M}$, for $i = 1, \ldots, n$.
- Then we have a natural map $\mathbb{Z}^n \to \tilde{M}$ by sending the unit basis vectors $e_i \to (i, v_i)$.
- This gives an exact sequence
  $0 \to R(A) \to \mathbb{Z}^n \to \tilde{M}$,
  which extends to
  $0 \to R(A)_{\mathbb{R}} \to \mathbb{R}^n \to \tilde{M}_{\mathbb{R}} \to 0$.

- The secondary polytope is a preimage of a point under the last map in the short exact sequence.
- Here, $R(A)$ is the secondary polytope, and we have $\dim R(A)_{\mathbb{R}} = n - d - 1$.
- We also wish to consider the dual sequence $0 \to \tilde{N}_{\mathbb{R}} \to \mathbb{R}^n \to R(A)^*_{\mathbb{R}} \to 0$.
- Duals of polytopes correspond to fern structures.
- Take a point $p \in R(A)^*_{\mathbb{R}}$, and consider the Hamiltonian, a smooth map from $\mathbb{C}^n$ to $\mathbb{R}^n$ given by $(z_1, \ldots, z_n) \to (|z_1|^2, \ldots, |z_n|^2)$, and compose it with the map to the dual space, and get...

- This composition is the momentum map μ_A:
  $\mu_A : \mathbb{C}^n \to R(A)^*_{\mathbb{R}}$.
- We factorize by the torus $\mu^{-1}(p)/T(A)$,
  where $T(A) = R(A)_{\mathbb{R}}/R(A)$ embeds into $U(1)^n$.
- The torus action respects the Hamiltonian map.
- This is known as *symplectic reduction* on this orbifold.
- So we have a quasi-smooth quasi-projective algebraic variety $X(p) = \mu^{-1}(P)/T(A)$.
- $0 \to R(A) \to \mathbb{Z}^n \to M \to 0$, where $T = R(A)_{\mathbb{R}}/R(A)$.

- Points $p \in I_{n+1}\sigma \iff$ vertices of Sec(A)
  $\iff$ convex triangulations of $\Delta =$Conv(A).
- $I_n\sigma$ is the corresponding secondary fan, dual to the polytope.
- Choose p so that the triangulation is unimodular.
- Then $X(p)$ is smooth, and Betti numbers are given by:
  $rk(H^i(X(p),\mathbb{Z})) = 0$ if $i = 2k+1$
  $rk(H^i(X(p),\mathbb{Z})) = \psi_k(\Delta)$ if $i = 2k$.
- Dimensions of Hodge filtration occur as Betti numbers.

- We now have two objects:
  $S_f^+ \approx PH^{d-1}(Z_f)$ and $H^*(X(p)$.
- Their dimensional components coincide.
- What is the relation? Quantum cohomology.
- Example in the case of curves: $d = 2$, and a lattice polytope equipped with a unimodular triangulation.
- What does it mean for $\mathbb{Z}_f \subset (\mathbb{C}^*)^2$ to be an affine curve?
- We associate every triangle with a sphere minus 3 points, and get out a multi-holed torus by gluing.

Example

- Consider a simplex with the above construction.
- This gives us a torus minus three points.
- So $\ell^*(\Delta) = g(\mathbb{Z}_f)$, $\dim H_1(\overline{Z_f}) = 2g$.
- We have $H \subset PH^d(Z_f)$, with $\dim H = \ell^*(\Delta)$.
- $\psi : S_f^* = (\ell(\Delta) - 3)t + \ell^*(\Delta)t^2$.
- This number is also the number of Kahler deformations.

- Consider again $X(p)$.
- $\dim H^2(X(p)) = \ell(\Delta) - 3$.
- $\dim H^n(X(p)) = \ell^*(\Delta)$.
- A better thing to consider is the function $\tilde{K}_q(X(p)) = K_0(X(p)) \oplus \mathbb{Z}$.
- This compares to $PH^1(Z_f)$.
- This worked only because of the unimodularity assumption on the triangulation.