• A lattice $\Delta \subset M_{\mathbb{R}}$.

• $P_\Delta(t) = \sum_{k \geq 0} \ell(k\Delta)t^k$.

• $\ell(k\Delta) = \#(M \cap k\Delta)$.

• $S_\Delta = \bigoplus_{k \geq 0} S^k_\Delta$.

• $P_\Delta(t) = \frac{\psi_\Delta(t)}{(1-t)^{d+1}}$, with $\deg \psi_\Delta(t) \leq d$. 
Example:

• $\Delta = [0, n]$.
• $\ell(k\Delta) = kn + 1$.
• $P_{\Delta}(t) = \sum_{k \geq 0} (kn + 1)t^k = \frac{1+(n-1)t}{(1-t)^2}$.
• Similarly, \( Q_\Delta(t) = \sum_{k>0} \ell^*(k \Delta) t^k \).

• \( \ell^*(k \Delta) = \#(\text{Int}(k \Delta) \cap M) \).

• \( Q_\Delta(t) = \frac{\phi_\Delta(t)}{(1-t)^{d+1}} \).

• \( \deg(\phi_{\Delta}) = d + 1 \).
Example:

- $\Delta = [0, n]$.
- $Q_\Delta(t) = \sum_{k>0} (kn - 1)t^k = \frac{t^2 + (n-1)t}{(1-t)^2}$.
- $\phi_\Delta(t)$, $\psi_\Delta(t)$ are related by $\phi_\Delta(t) = t^{d+1}\psi_\Delta(t^{-1})$.
- If $\phi_\Delta(t) = \sum \phi_i(\Delta)t^i$ and $\psi_\Delta(t) = \sum \psi_j(\Delta)t^j$, then $\psi_i(\Delta) = \phi_{d+1-i}(\Delta)$ for all $0 \leq i \leq d$.
- $\psi_0(\Delta) = 1 = \phi_{d+1}(\Delta)$.
- $\psi_d(\Delta) = \phi_1(\Delta) = \ell^*(\Delta)$.
• Behind all this is the graded Artinian ring 
\[ S_f = S_\Delta / \langle F_0, \ldots, f_d \rangle S_\Delta. \]

• The Poincare series gives \( P(S_f, t) = \phi_\Delta(t) \), and 
\[ \dim(S^i_f) = \psi_i(\Delta). \]

• We constructed \( (\text{Int}(C_\Delta) \cap \tilde{M}) = I_\Delta \subset S_\Delta. \)

• \( I_\Delta \) is a canonical ideal.

• Similarly, we consider \( I_f = I_\Delta / \langle F_0, \ldots, F_d \rangle I_\Delta. \)

• \( Q_\Delta(t) \) comes from \( I_\Delta. \)
• $\phi_\Delta(t) = (1 - t)^{d+1} Q_\Delta(t)$.

• $\dim I_k^f = \phi_k(\Delta)$.

• We get a multiplication $S_f \times I_f^{d+1-i} \rightarrow I_f^{d+1} \approx K$.

• We get a duality $S_f^i \approx (I_f^{d+i-1})^*$. 
Cohomology.

- We are interested in $H^i(Z_f)$ and $H^i_c(Z_f)$ for $Z_f \subset \mathbb{T}^d \approx (\mathbb{C}^*)^d$.
- Assume $d \geq 2$, since $d = 1$ is uninteresting.
- Bottom and top cohomology: 
  
  \[ H^0(Z_f) \approx \mathbb{Z} \text{ and } H^2_c((d-1)Z_f) \approx \mathbb{Z}. \]
  
  \[ H^i(Z_f) \times H^2_c(2d-2-i)(Z_f) \rightarrow H^2_c(2(d-1))(Z_f) \approx \mathbb{Z}. \]
- Reminiscent of the case with $S_f$ and $I_f$. 
• Take $Z_f \subset (\mathbb{C}^*)^d$. Then
\[ \sum_i (-1)^i \dim H^i(Z_f) = (-1)^{d-1} \text{Vol}(\Delta) = d! \text{Volume}(\Delta). \]
• Recall $\dim S_f = \dim I_f = \text{Vol}(\Delta)$.
• Lefschetz Theorem: For $Z_f \subset \mathbb{T}^d$, the restriction
\[ H^i(\mathbb{T}^d) \rightarrow H^i(Z_f) \] is an isomorphism for $0 \leq i \leq d - 2$, and injective for $i = d - 1$.
• Also, $H^i(\mathbb{T}^d)$ is the exterior algebra.
• $Z_f$ is affine, so $H^i(Z_f) = 0$ for $i \geq d$. 
• Middle Cohomology: $\dim H^{d-1}(Z_f) = \text{Vol}(\Delta) + d - 1$.

• So the restriction $H^{d-1}(\mathbb{T}^d) \to H^{d-1}(Z_f)$ is injective.

• The primitive part $\dim PH^{d-1}(Z_f) = \text{Vol}(\Delta) - 1$ is the cokernel of the above restriction.
Mixed Hodge Structure:

- We consider $H^i(X)$ and $H^i_c(X)$.

- Deligne: This cohomology has 2 filtrations, the Hodge filtration and the weight filtration, giving the Hodge components or Hodge numbers, $H^{p,q}(X)$.

- Example: $d=2$, $\mathbb{Z}_f \subset (\mathbb{C}^*)^2$ an affine curve.

- The toric compactification $\overline{Z}_f$ is smooth and projective, so topologically looks like a torus.

- Each side of the corresponding polytope has some lattice points.
• \( \#(\overline{Z_f} \setminus Z_f) = \#(\partial \Delta \cap M) \). 

• Table of ranks:
  \( H^0(\overline{Z_f}) \) has rank 1.
  \( H^1(\overline{Z_f}) \) has rank 2\( g \).
  \( H^2(\overline{Z_f}) \) has rank 1.

• Thus the Euler number is \( e(\overline{Z_f}) = 2 - 2g \).
• If we remove one point, we don’t change the top degree.

• So $H^2(Z_f) = 0$, and
  $\dim H^1(Z_f) = \dim H^1(\overline{Z_f}) + (\#\partial\Delta \cap M - 1)$.

• $\dim H^0(Z_f) = 1$.

• This agrees with $\dim H^{d-1}(Z_f) = \text{Vol}(\Delta) + (d - 1)$,
  since we have $d = 2$, and
  $2g + (\partial \cap M) - 1 = \dim H^i(Z_f) = \text{Vol}(\Delta) + 1$.

• The genus is computed by $g = \#\ell^*(\Delta)$.
Example:

- For $H^1(Z_f)$, we consider the Hodge pieces:

\[
\begin{array}{|c|c|}
\hline
\text{g} & \#(\partial \Delta \cap M) - 1 \\
\hline
0 & \text{g} \\
\hline
\end{array}
\]

Here, horizontal filtration is Hodge, diagonal is weight.

- Similarly for $H^1_c(Z_f)$, we consider the Hodge pieces:

\[
\begin{array}{|c|c|}
\hline
\text{g} & 0 \\
\hline
\#(\partial \Delta \cap M) - 1 & \text{g} \\
\hline
\end{array}
\]

Here, horizontal filtration is Hodge, diagonal is weight.

- So $h^{1,1} = \#(\partial \Delta \cap M) - 1$ is non-trivial.
\begin{itemize}
\item $h^{1,0} + h^{1,1} = \ell^*(\Delta) + \partial \Delta \cap M - 1 = S_f^1$.
\item $h^{0,0} + h^{0,1} = 0 + \ell^*(\Delta) = \dim S_f^2$.
\item For any polytope $\Delta$ of dimension $d$,

$$\text{Gr}_H \text{PH}^{d-1}(Z_f) \approx S_f^+ \subset S_f.$$

\item Also, $\text{Gr}_H \text{PH}_c^{d-1}(Z_f) \approx \bigoplus_{i=1}^d I_f$
\end{itemize}
General Statement:

- $H^*(z_f) \leftarrow H^*(\mathbb{T}^d) \cong \Lambda^* M$.
- Koszul complex $F_0, F_1, \ldots, F_d$ in $S_{\Delta}$.
- We now considered a twisted version:
  $\Delta \subset M_{\mathbb{R}} \supset M \subset M^1 \subset M_{\mathbb{Q}}$.
- $f(x) = \sum_{m \in A} a_m x^m$.
- Etale covering: $Z^1_f \to Z_f$ of degree $[M' : M]$. 