- A lattice $\Delta \subset M_{\mathbb{R}}$.
- $P_\Delta(t) = \sum_{k \geq 0} \ell(k\Delta) t^k$.
- $\ell(k\Delta) = \#(M \cap k\Delta)$.
- $S_\Delta = \bigoplus_{k \geq 0} S_\Delta^k$.
- $P_\Delta(t) = \frac{\psi_\Delta(t)}{(1-t)^{d+1}}$, with $\deg \psi_\Delta(t) \leq d$.

Example:

- $\Delta = [0, n]$.
- $\ell(k\Delta) = kn + 1$.
- $P_\Delta(t) = \sum_{k \geq 0} (kn+1) t^k = \frac{1+(n-1)t}{(1-t)^2}$.

- Similarly, $Q_\Delta(t) = \sum_{k>0} \ell^*(k\Delta)t^k$.
- $\ell^*(k\Delta) = \#(\mathrm{Int}(k\Delta) \cap M)$.
- $Q_\Delta(t) = \frac{\phi_\Delta(t)}{(1-t)^{d+1}}$.
- $\deg(\phi_{Delta}) = d + 1$.

Example:

- $\Delta = [0, n]$.
- $Q_\Delta(t) = \sum_{k>0}(kn-1)t^k = \frac{t^2+(n-1)t}{(1-t)^2}$.
- $\phi_\Delta(t)$, $\psi_\Delta(t)$ are related by $\phi_\Delta(t) = t^{d+1}\psi_\Delta(t^{-1})$.
- If $\phi_\Delta(t) = \sum \phi_i(\Delta)t^i$ and $\psi_\Delta(t) = \sum \psi_j(\Delta)t^j$, then $\psi_i(\Delta) = \phi_{d+1-i}(\Delta)$ for all $0 \leq i \leq d$.
- $\psi_0(\Delta) = 1 = \phi_{d+1}(\Delta)$.
- $\psi_d(\Delta) = \phi_1(\Delta) = \ell^*(\Delta)$.

- Behind all this is the graded Artinian ring $S_f = S_\Delta / \langle F_0, \ldots, f_d \rangle S_\Delta$.
- The Poincare series gives $P(S_f, t) = \phi_\Delta(t)$, and $\dim(S_f^i) = \psi_i(\Delta)$.
- We constructed $(\mathrm{Int}(C_\Delta) \cap \tilde{M}) = I_\Delta \subset S_\Delta$.
- I_Δ is a canonical ideal.
- Similarly, we consider $I_f = I_\Delta / \langle F_0, \ldots, F_d \rangle I_\Delta$.
- $Q_\Delta(t)$ comes from I_Δ.

- $\phi_\Delta(t) = (1-t)^{d+1} Q_\Delta(t)$.
- $\dim I_f^k = \phi_k(\Delta)$.
- We get a multiplication $S_f \times I_f^{d+1-i} \to I_f^{d+1} \approx K$.
- We get a duality $S_f^i \approx (I_f^{d+i-1})^*$.

Cohomology.

- We are interested in $H^i(Z_f)$ and $H_c^i(Z_f)$ for $Z_f \subset \mathbb{T}^d \approx (\mathbb{C}^*)^d$.
- Assume $d \geq 2$, since $d = 1$ is uninteresting.
- Bottom and top cohomology:
  $H^0(Z_f) \approx \mathbb{Z}$ and $H_c^{2(d-1)}(Z_f) \approx \mathbb{Z}$.
- $H^i(Z_f) \times H_c^{2d-2-i}(Z_f) \to H_c^{2(d-1)}(Z_f) \approx \mathbb{Z}$.
- Reminiscent of the case with S_f and I_f.

- Take $Z_f \subset (\mathbb{C}^*)^d$. Then
  $\sum_i (-1)^i \dim H^i(Z_f) = (-1)^{d-1}\mathrm{Vol}(\Delta) = d!\mathrm{Volume}(\Delta)$.
- Recall $\dim S_f = \dim I_f = \mathrm{Vol}(\Delta)$.
- Lefschetz Theorem: For $\mathbb{Z}_f \subset \mathbb{T}^d$, the restriction
  $H^i(\mathbb{T}^d) \to H^i(Z_f)$ is an isomorphism for $0 \le i \le d-2$,
  and injective for $i = d-1$.
- Also, $H^i(\mathbb{T}^d)$ is the exterior algebra.
- Z_f is affine, so $H^i(Z_f) = 0$ for $i \ge d$.

- Middle Cohomology: $\dim H^{d-1}(Z_f) = \mathrm{Vol}(\Delta) + d - 1$.
- So the restriction $H^{d-1}(\mathbb{T}^d) \to H^{d-1}(Z_f)$ is injective.
- The primitive part $\dim PH^{d-1}(Z_f) = \mathrm{Vol}(\Delta) - 1$ is the cokernel of the above restriction.

Mixed Hodge Structure:

- We consider $H^i(X)$ and $H^i_c(X)$.
- Deligne: This cohomology has 2 filtraions, the Hodge filtration and the weight filtration, giving the Hodge components or Hodge numbers, $H^{p,q}(X)$.
- Example: d=2, $\mathbb{Z}_f \subset (\mathbb{C}^*)^2$ an affine curve.
- The toric compactification $\overline{Z_f}$ is smooth and projective, so topologically looks like a torus.
- Each side of the corresponding polytope has some lattice points.

- $\#(\overline{Z_f} \backslash Z_f) = \#(\partial\Delta \cap M)$.
- Table of ranks:
  $H^0(\overline{Z_f})$ has rank 1.
  $H^1(\overline{Z_f})$ has rank $2g$.
  $H^2(\overline{Z_f})$ has rank 1.
- Thus the Euler number is $e(\overline{Z_f}) = 2 - 2g$.

- If we remove one point, we don't change the top degree.
- So $H^2(Z_f) = 0$, and
  $\dim H^1(Z_f) = \dim H^1(\overline{Z_f}) + (\#\partial\Delta \cap M - 1)$.
- $\dim H^0(Z_f) = 1$.
- This agrees with $\dim H^{d-1}(Z_f) = \mathrm{Vol}(\Delta) + (d-1)$,
  since we have $d = 2$, and
  $2g + (\partial \cap M) - 1 = \dim H^i(Z_f) = \mathrm{Vol}(\Delta) + 1$.
- The genus is computed by $g = \#\ell^*(\Delta)$.

Example:

- For $H^1(Z_f)$, we consider the Hodge pieces:

  $H^1(Z_f)$:

g	$\#(\partial\Delta \cap M) - 1$
0	g

  Here, horizontal filtration is Hodge, diagonal is weight.

- Similarly for $H^1_c(Z_f)$, we consider the Hodge pieces:

  $H^1_c(Z_f)$:

g	0
$\#(\partial\Delta \cap M) - 1$	g

  Here, horizontal filtration is Hodge, diagonal is weight.

- So $h^{1,1} = \#(\partial\Delta \cap M) - 1$ is non-trivial.

- $h^{1,0} + h^{1,1} = \ell^*(\Delta) + \partial\Delta \cap M - 1 = S_f^1$.
- $h^{0,0} + h^{0,1} = 0 + \ell^*(\Delta) = \dim S_f^2$.
- For any polytope Δ of dimension d,
  $\mathrm{Gr}_H PH^{d-1}(Z_f) \approx S_f^+ \subset S_f$.
- Also, $\mathrm{Gr}_H PH_c^{d-1}(Z_f) \approx \bigoplus_{i=1}^d I_f$

General Statement:

- $H^*(z_f) \leftarrow H^*(\mathbb{T}^d) \approx \Lambda^* M$.
- Koszul complex $F_0, F_1, \ldots, F_d$ in S_Δ.
- We now considered a twisted version:
  $\Delta \subset M_\mathbb{R} \supset M \subset M^1 \subset M_\mathbb{Q}$.
- $f(x) = \sum_{m \in A} a_m x^m$.
- Etale covering: $Z_f^1 \to Z_f$ of degree $[M' : M]$.