Instanton Sums and Monodromy

Lecture 3

$\rho : G \to U(1)^n$, $W(x_1, \ldots, x_n)$ G-invariant.

Example 1

$G = U(1)$

x_0	x_1	x_2	$\cdots$	x_5
-5	1	1	$\cdots$	1

$$W(x_0, \ldots, x_n) = x_0 P_5(x_1, \ldots, x_5)$$

Example 2

$G = U(1)^2$. $\left(\mathrm{Bl}_{\mathrm{sing}}\, \mathbb{P}^{1,1,2,2,2} \right)$

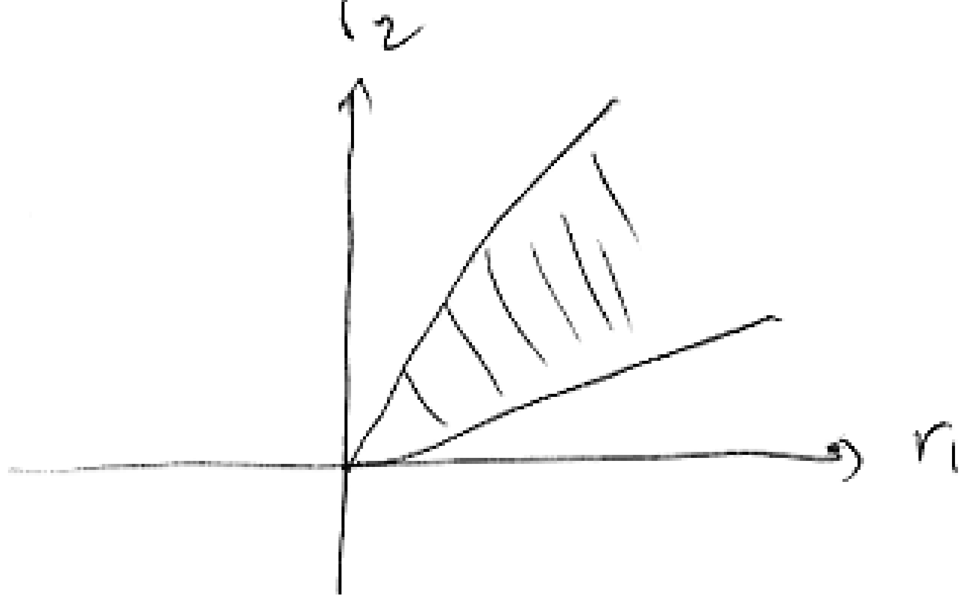
x_0	x_1	x_2	x_3	x_4	x_5	x_6
-4	0	0	1	1	1	1
0	1	1	0	0	0	-2

$$W(x_0, \dots, x_n) = x_0\, \underbrace{P(x_1, \dots, x_6)}_{G\text{-invariant}}$$

$\mu^{-1}(r)/G.$

$r \in \mathfrak{g}^*$

example 2



topology of $\mu^{-1}(r)/G$,
set of coordinate charts

$\Big\}$ constant within regions.

$$r_1 = -4|x_0|^2 + |x_3|^2 + |x_4|^2 + |x_5|^2 + |x_6|^2$$

"<u>Secondary fan</u>" is the fan structure on $\mathfrak{g}^*$.

$$\mu^{-1}(r)/G \cong \mathbb{C}^n \setminus Z(r) / G_{\mathbb{C}}$$

$$Z(r) = \bigcup_{I \in \mathcal{A}(r)} \bigcap_{i \in I} \{x_i = 0\}$$

$$I \subset \{1, \ldots, n\}$$

$I \in \mathcal{A}(r) \iff \{x_i\}_{i \in I}$ is not a subset of coordinates in any coordinate.

One feature from physics:

1-loop correction,

complete determines location of

singular locus.

parameters $\begin{cases} \tau \in \mathrm{Hom}(g, \mathbb{C}) \\ \text{coefs of } W \end{cases}$

not relevant

bosonic potential:

$$U = \frac{1}{2e^2}\sum (D_a)^2 + \sum \left|\frac{\partial W}{\partial \varphi_i}\right|^2$$
$$+ 2\sum \bar{\sigma}_a \sigma_b \sum Q_i^a Q_i^b |\varphi_i|^2$$

bases V_a of gauge fields

$$M^{2|(2,2)} \to \mathfrak{g}$$

$\Sigma_a = \bar{D}_+ D_- V_a$

σ_a = lowest component $\Sigma_a|_{\theta^2}$.

For generic values of $|q_i|$'s (of parameters)

we get pos def Hermitian form on the space of σ's

to minimize, $\sigma_a = 0$.

For special values, pos. semi-definite.

Divergence

regulated with a cutoff:

$\to$ effective description of theory in σ's.

conclusion about parameter values:

$$2\pi i \tau_a = \sum_{i=1}^{n} Q_i^a \log\left(\sum Q_i^b \sigma_b\right)$$

(assoc. to group G)

...

can be singularities assoc. to subgroups $G' \subseteq G$.

~~Example 1~~

$$q_a = e^{2\pi i \tau_a} = \prod_{i=1}^{n} \left(\sum Q_i^b \sigma_b \right)^{Q_i^a}$$

Example 1

x_0	x_1	$\cdots$	x_5
-5	1	$\cdots$	1

$$q = (-5\sigma)^{-5}\sigma^1\sigma^1\sigma^1\sigma^1\sigma^1 = (-5)^{-5}$$

Predicted singularity of physics at

$$q = (-5)^{-5}.$$

Correlation function (last time)

$$\frac{5}{1+5^5 q}.$$

Example 2

σ)

x_0	x_1	x_2	x_3	x_4	x_5	x_6
-4	0	0	1	1	1	1
0	1	1	0	0	0	-2

$$q_1 = (-4\sigma_1)^{-4}(\sigma_1)^1(\sigma_1)^1(\sigma_1)^1(\cancel{\sigma_1})^1(\sigma_1-2\sigma_2$$

$$q_2 = (\sigma_2)^1(\sigma_2)^1(\sigma_1-2\sigma_2)^{-2}$$

$$q_1 = (-4)^{(-4)} \left(\frac{\sigma_1 - 2\sigma_2}{\sigma_1} \right) = (-4)^{-4} (1-2$$

$$q_2 = \frac{\sigma_2^{\,2}}{(\sigma_1 - 2\sigma_2)^2} = \frac{(\sigma_2/\sigma_1)^2}{\left(1 - 2\frac{\sigma_2}{\sigma_1}\right)^2}$$

$$\frac{\sigma_2}{\sigma_1} = \frac{1-4^4 q_1}{2}.$$

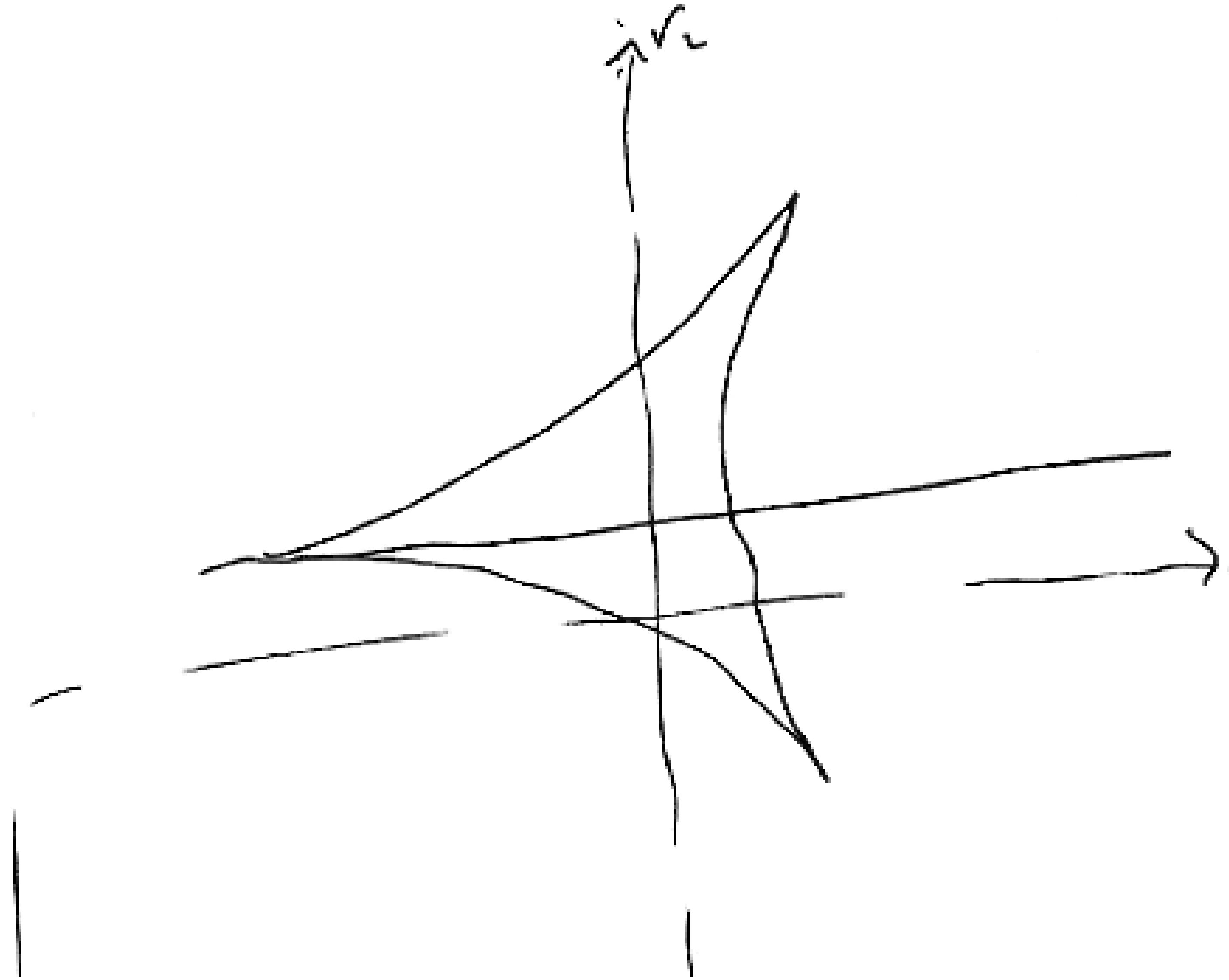
$$q_2 = \frac{\left(\frac{1-4^4 q_1}{2}\right)^2}{\left(4^4 q_1\right)^2}$$

$$\boxed{4^9 q_1^2 q_2 = (1-4^4 q_1)^2}$$

$G' = (\text{gen. by } \exp g_2)$

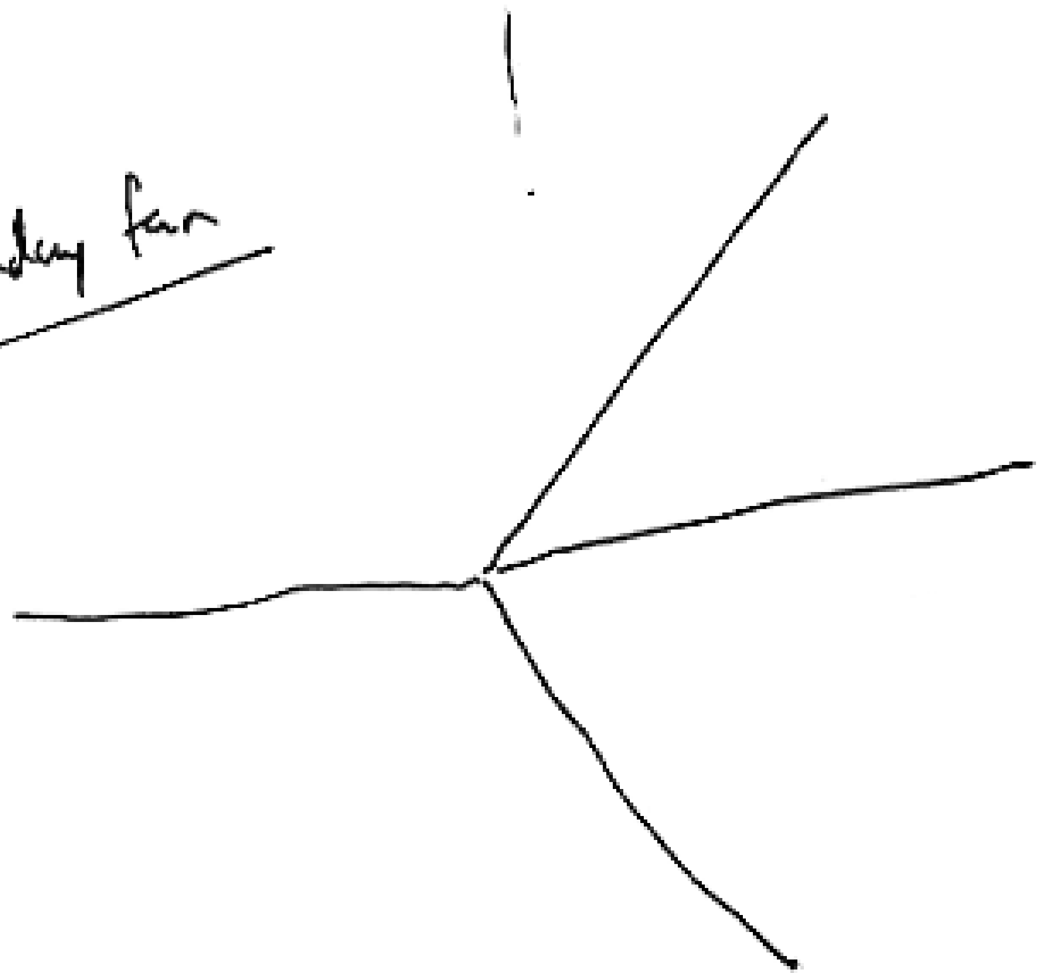
$$q_2 = \sigma_\phi^1 \sigma_\phi^2 (-2\sigma)^{-2} = (-2)^{-2}.$$

$$\boxed{q_2 = 1/4}$$



r_2
r_1

Secondary fan



Instanton sums

Instanton moduli spaces.

$x_i = f_i(s,t)$ ← homog

$\mathbb{C}^2$

$\mathbb{CP}^1$

$\deg f_i = d_i$.

Labeling:

n_a, $a = 1, \ldots, \dim \mathfrak{g}$.

$$e^{2\pi i \sum z_a n_a} = \prod q_a^{n_a}$$

$$\deg f_i = \sum Q_i^a n_a = d_i.$$

<u>Requirement</u>: $\vec{\eta} \in$ dual cone to the cone in 2nd fan in which r lives.

$X_i \longleftrightarrow f_i(s,t)$ of degree d_i

$$f_i(s,t) = \sum_{j=0}^{d_i} f_i^{(j)} s^j t^{d_i - j}$$

$\{f_i^{(j)}\}$ are homogeneous coords on instanton moduli space.

G acts on $f_i^{(j)}$ as it acts on X_i.

$G_{\mathbb{C}}$ acts in same way.

$$X = \mathbb{C}^n \setminus Z(\Gamma) / G_{\mathbb{C}}.$$

$$Z(\Gamma) = \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} \{x_i = 0\}$$

$$Z(\Gamma)_{\vec{n}} = \bigcup_{I \in \mathcal{I}} \bigcap_{i \in I} \{f_i^{(j)} = 0\}$$

$$\mathcal{M}_{\vec{n}} = \mathbb{C}^{n(\vec{n})} \setminus Z(r)_{\vec{n}} / G_{\mathbb{C}}.$$

$$= \mu_{\vec{n}}^{-1}(r) / G.$$

where $\mu_{\vec{n}} : \mathbb{C}^{n(\vec{n})} \to \mathfrak{g}^*$

replace $|X_i|^2$ with $\sum |f_i^{(j)}|^2$.

to

Provided that $d_i \geq 0$, get ~~the~~ contributions to correlation function:

$$\langle \eta_{i_1} \eta_{i_2} \eta_{i_3} (-K)^{-d_0+1} x^{\vec{n}} \rangle_{\vec{n}}$$

$\chi_{\vec{n}}$ only need if $d_i < 0$ for some i.

$$\chi_{\vec{n}} = \prod_{d_i < 0} (\xi_i)^{-d_i - 1}.$$

$$\xi_i^{(j)} = c_1(\{f_i^{(j)} = 0\})$$

this coho class is indpt of j.

<u>linear relations</u>

$$\cancel{\eta_a = \sum Q_i^a \xi_i}$$

$$\xi_i = \sum Q_i^a \eta_a$$

nonlinear relations

previous: $\xi_{i_1} \cdots \xi_{i_k} = 0$

new: $(\xi_{i_1})^{d_{i_1}+1} \cdots (\xi_{i_k})^{d_{i_k}+1} = 0$

Example 2

$$X_j^{\vec{n}} = \left\langle \eta_1^{3-j}\, \eta_2^{j}\, K^{4n_1+1}\, x_{\vec{n}} \right\rangle_{\vec{n}}$$

$$(K = -4\eta_1).$$

$$X_j^{\vec{n}} = 4^{4n_1+1} \left\langle \eta_1^{4n_1+4-j}\, \eta_2^{j} \right\rangle_{\vec{n}}.$$

Using relations

$$(\xi_1)^{d_1+1}(\xi_2)^{d_2+1} = \eta_2^{d_1+d_2+2} = 0$$

$$(\xi_3)^{d_3+1}(\xi_4)^{d_4+1}(\xi_5)^{d_5+1}(\xi_6)^{d_6+1} = \eta_1^{d_3+d_4+d_5+3}(\eta_1-2\eta_2)^{d_6+1}$$

and some combinatorics.

$$X_j = \sum_{\substack{n_1 \geq 0 \\ n_2 \geq 0}} 2^{8n_1 + 2n_2 + 3 - j} \binom{n_1 + 1 - j}{2n_2 + 1 - j} q_1^{n_1} q_2^{n_2}$$

$$X_0 = \frac{8}{\Delta} \qquad \Delta = (1 - 2^8 q)^2 - 2^{18} q_1^2 q_2$$

$$X_1 = \frac{4(1 - 2^8 q_1)}{\Delta}$$

$$X_2 = \frac{8q_2(2^9 q_1 - 1)}{(1-4q_2)\Delta}$$

$$X_3 = \frac{\qquad}{(1-4q_2)^2\,\Delta}$$

Mirror symmetry

$$\rho : G \to U(1)^n, \quad W.$$

$$\rightsquigarrow \quad \hat{\rho} : \hat{G} \to U(1)^{\hat{n}}, \quad \hat{W}$$

Another set of correlation functions:
related to VHS determined by $\widehat{W} = 0$.

discriminant locus for $\widehat{W}$, as a function of parameters, is

$$\Delta (1-4q_2)^2 \quad \text{in example 2.}$$

(Γ-description $\longleftrightarrow$ "Horn uniformization")

Kapranov