Fact Let $K/_{\mathbb{Q}}$ be a function field then one has:

1.
$$\mathbf{td}(K) = d \iff \begin{cases} \forall\, t_1, \dots, t_{d+2} \implies q_{(t_1,\dots,t_{d+2})} \\ \qquad\qquad \text{is universal over } K(\sqrt{-1}) \\ \textbf{and} \\ \exists\, t_1, \dots, t_{d+2} \text{ such that } q_{(t_1,\dots,t_{d+2})} \\ \qquad\qquad \text{does not represent 0 over } K(\sqrt{-1}) \end{cases}$$

2. Suppose $\mathbf{td}(K) = d$ and $t_1, \dots, t_d \in K^\times$, $\exists : a$, b rational numbers such that $q_{(t_1,\dots,t_d,a,b)}$ does not represent 0 over $K(\sqrt{-1})$. Then $t_1, \dots, t_d$ is a transcendence base

Zusatz $\forall\, t_1, \dots, t_d$ TB $\exists$ "many" natural numbers $a_1, \dots, a_d, a, b$ such that $q_{(t_1-a_1,\dots,t_d-a_d,a,b)}$ satisfies 2).

<u>Proof</u> **Step** 1 Cohomological interpretation of $\mathbf{td}(K) = d$

Lemma Let $E/_K$ be a finite extension with $\sqrt{-1} \in E$. Then the following are equivalent:

1. $\mathbf{td}(K) = d$
2. $\forall\, t_1, \ldots, t_{d+3} \in K^\times \implies (t_1) \cup \cdots \cup (t_{d+3}) = 0$ in $\mathbf{H}^{d+3}(E, \mu_2)$

   and

   $\exists\, t_1, \ldots, t_{d+2} \in K^\times$ such that $(t_1) \cup \cdots \cup (t_{d+2}) \neq 0$ in $\mathbf{H}^{d+2}(E, \mu_2)$

Idea of Proof

- Use: let k be a non-real number field, and $K/_k$ a function field then $\mathbf{cd}_2\,(K) = 2 + \mathbf{td}(K/_k)$
- Now this assertion is equivalent to 2)
- Based on:
  - (Tate) $\mathbf{cd}_2\,(k) = 2$
  - (Serre, Galois Cohomology) In general $K/_k$ function field $\implies$ $\mathbf{vcd}_2\,(K) = \mathbf{vcd}_2\,(k) + \mathbf{td}(K_k)$

Step 2

- **Milnor Conjecture** E field, **Char**$(E) \neq 2$
- Milnor $\mathbf{K}$ groups: $\mathbf{K}^m(E)$ generated by symbols $\{a_1, \ldots, a_n\}$ satisfying relations
- Witt Ring of E: $\mathbf{W}(E) = \{$ all anisotropic quadratic forms over E $\}$
  Together with addition and multiplication
- $I(E) \subseteq \mathbf{W}(E)$ ideal of even dimensional quadratic forms

- $I^2(E)$, $I^3(E)$, ...
-

$$\mathbf{Gr}(\mathbf{W}(E)) = \bigoplus_{n=0}^{\infty} I^n(E)/_{I^{n+1}(E)}$$

- $I^0(E)/_{I(E)} = \mathbb{Z}/_{2\mathbb{Z}} \xleftarrow{\text{degree } (\mathbf{mod}\ 2)} \mathbf{W}(E)$
- $I^1(E)/_{I^2(E)} = K^\times/_q \xleftarrow{\text{discriminant}} \mathbf{W}(E)$
- $I^2(E)/_{I^3(E)} = \xleftarrow{\text{Clifford invariant}} \mathbf{W}(E)$
- **Fact** $I^n(E)$ is generated by the n-fold Pfister forms

- **The famous triangle** $\forall\, n \geq 0$ (Milnor, Tate...)

$$\mathbf{K}^n\, E/_2 \xrightarrow{\quad h_n \quad} \mathbf{H}^n(E, \mu_2)$$

$$\{a_1, \dots, a_n\} \overset{s_n}{\longmapsto} (a_1) \cap \cdots \cap (a_n)$$

$$q_{(-a_1, \dots, -a_n)} \overset{e_n}{\dashrightarrow}$$

$$I^n(E)/_{I^{n+1}(E)}$$

- **Milnor Conjecture** h_n, s_n are isomorphisms. In particular e_n exists
- Long history
- (Voevodsky, et.al.) MC is OK

Lemma Let E be as above, and $q_{(a_1,\ldots,a_n)}$ n-fold Pfister form. Then the following are equivalent:

1) q represents $a \in E^\times$

1′) $q_{(a_1,\ldots,a_n,a)}$ is hyperbolic

2) $(-a_1) \cap (-a_2) \cdots \cap (-a_n) \cap (-a) \in \mathbf{H}^{n+1}(E, \mu_2)$ is trivial

Relies on Milnor Conjecture

back to the proof of the fact

- For 1), combine Lemmas above
- for 2), **Claim** $t_1, \ldots, t_d$ are algebraically independent
  - Proof by contradiction.
  - Let K_o =Relative closure of $\mathbb{Q}(t_1, \ldots, t_d)$ in K
  - $\implies \mathbf{td}(K_o) < d$
  - The fact that $q_{(t_1,\ldots,t_d,a,b)}$ does not represent 0 over $K(\sqrt{-1})$, thus over $K_o(\sqrt{-1})$ contradicts 1)

For the **Zusatz**

- $R = \mathbb{Z}[t_1, \ldots, t_d]$
- S =integral closure of R in $K(\sqrt{-1})$
- Higher dimensional Cebotarev density theorem:

  $\exists$ "many" closed points x of R which are totally split in S
- let $y \in S$
- $\hat{\mathcal{O}}_{R,x} \cong \hat{\mathcal{O}}_{S,y} \cong \mathbb{Z}_q[\![t_1 - a_1, \ldots, t_d - a_d]\!]$

Folklore

- K some field, I index set (e.g. ω_o)
- $\mathcal{D}$ is an ultrafilter
- $K_{\mathcal{D}} = K^I/_{\mathcal{D}}$
- **Question** analyze/describe the relatively algebraically closed subfields $K \subset L \subset K_{\mathcal{D}}$

Proposition(Folklore) For L as above: $\forall\, l_i \subset L$ finitely generated over K one has:

- Let $X_i \longrightarrow K$ be some model, then $X_i(K)$ is Zariski dense.
- Conversely, if $Y|_K$ is some variety the $K(Y) \hookrightarrow K_{\mathfrak{D}} \Longleftrightarrow Y(K)$ is Zariski dense

Theorem(Faltings) K an number field. For every $L \subset K_{\mathfrak{D}}$ as above with $\mathbf{td}(L/_K) = 1$ then every L_i is the function field of some projective smooth curve $X_i \longrightarrow K$ with $g_{X_o} \leq 1$

Conjectural Theorem Suppose the Lang conjectures are true, $L \subset K_{\mathfrak{D}}$ with $\mathbf{td}(L/_K)$ finite and $L_i \subset L$ finitely generated, then Li is not of general type