k = base field $\quad \dfrac{\text{arithmetic}}{\text{prime field}} \quad / \quad \dfrac{\text{geometric}}{\text{algebraically closed}}$

$$K$$

| finite

$$k(t_1, \dots, t_d)$$

| purely transcendental

$$k$$

Detecting **td**(K/k)

- **Idea** Use homogeneous forms
- $q(x_1, \ldots, x_n)$ homogeneous form over some field K
- $K'/_K$ field extension
- **Definition**
  - q is universal over K' if the following $\forall\exists$ formula is true in K':

$$\forall a \in {K'}^{\times} \exists a_1, \ldots, a_n \in K' \text{ such that } q(a_1, \ldots, a_n) = a$$

  - q represents zero over K' if the following $\exists$ formula is true in K':

$$\exists a_1, \ldots, a_n \in K' \text{ not all zero, such that} q(a_1, \ldots, a_n) = 0$$

Special Case Suppose that $K' = K[\alpha]$ for α an algebraic integer over K.

Theorem

- q is universal over K' is a $\forall\exists$ formula with only parameters the coefficients q over K
- q does not represent zero over K' is a $\forall$ formula with only parameter the coefficients of q over K

How To Use It

- $(t_1, \ldots, t_r)$ a system of elements in $K^\times$
- p a natural number
- Consider $(\underline{t}_{\underline{i}})_{\underline{i}}$ where $\underline{t}_{\underline{i}} = t_1^{i_1} \cdots t_r^{i_r}$ $\forall 0 \leq i_1 \ldots i_r < p$
- Set $q^{(p)}_{(t_1,\ldots,t_d)} = \sum_{\underline{i}} \underline{t}_{\underline{i}} x^p_{\underline{i}}$
- Coefficients depend only on $(t_1, \ldots, t_r)$
- p^r variables

- **Consequence** $q^{(p)}_{(t_1,\ldots,t_d)}$
  - is universal over $K' = K[\alpha]$
  - does not represent 0 over $K' = K[\alpha]$
- **Application** $\exists(t_1, \ldots, t_d)$ such that

$$q^{(p)}_{(t_1,\ldots,t_d)} \begin{cases} \text{is universal } \ldots\ \exists\forall \text{ sentence} \\ \text{does not represent } 0 \ldots\ \exists\forall \text{ sentence} \end{cases}$$

<u>Case 1</u> Char $k = p > 0$

- Recall p-Bases of $K/_k$
- Let k be perfect, $K/_k$ function field
- Then for a system $(t_1, \ldots, t_d)$ of elements in $K^\times$ the following are equivalent:
  1. $(t_1, \ldots, t_d)$ is a p-Basis for K
  2. $q^{(p)}_{(t_1,\ldots,t_d)}$ is universal but does not represent 0
  3. $dt_1, \ldots, dt_d$ is a K-basis of $\Omega_{K/_k}$

Fact (**Char** $k = p > 0$) $K/_k$ arithmetic/geometric function field

1. $\mathbf{td}(K/_k) = d \iff \exists\forall\exists$ sentence:

   $$\exists(t_1, \ldots, t_d) \text{ such that } q^{(p)}_{(t_1,\ldots,t_d)}$$
   $$\text{is universal but does not represent } 0$$

2. $(t_1, \ldots, t_d)$ is a separable transcendence basis $\iff$

   1) holds for $(t_1, \ldots, t_d)$

- Preparation for geometric + **Char** $k \neq 2$ and arithmetic case
- Context as above $p = 2$ then $q_{(t_1,\ldots,t_r)} := q^{(2)}_{(t_1,\ldots,t_r)}$ is the r-fold Pfister form defined by $(t_1, \ldots, t_r)$

Lemma

- Let R be a discrete valued ring (DVR)
- π uniformizing parameter
- $k = R/_{(\pi)}$
- Let $\overline{q}_o = \overline{q}_o(x_1, \ldots x_n)$ be a diagonal quadratic form over k
- $\overline{q}$ a lifting of $\overline{q}_o$ over R
- $q := q_o???\underbrace{(x_o^2 + \pi x_1^2)}_{q(\pi)} = q(x_1, \ldots, x_n) + \pi q(y_1, \ldots, y_n)$
- If $\overline{q}_o$ does not represent 0 over k then q does not represent 0 over $K = \mathbf{Qout}(R)$

Proof Exercise

Fact (k algebraically closed, **Char** $\neq 2$)

$K/_k$ a function field then

1. $\mathbf{td}(K/_k) = d \iff \forall\, t_1, \ldots, t_d \in K^\times$, $q_{(t_1,\ldots,t_d)}$ is universal **and** $\exists\, t_1, \ldots, t_d \in K^\times$ such that $q_{(t_1,\ldots,t_d)}$ does not represent 0
   (Thus $\mathbf{td}(K/_k)$ is a $\forall\exists \wedge \exists\forall\exists$ sentence)
2. Moreover, if $(t_1, \ldots, t_d)$ is a separable transcendence base and $\mathbf{td}(K/_k) = d$ then $\exists\, a_1, \ldots, a_d \in k$ such that $q_{(t_1-a_1,\ldots,t_d-a_d)}$ does not represent 0

Proof

- Use the C_r-property of fields
- $\mathcal{K}$ is a C_r-field if all homogeneous forms $q(x_1, \ldots, x_n)$ of degree d represent 0 provided $n > d^r$
- **Example** (Chevellay-Waring) Finite fields are C_1
- **Example** if K is complete, discrete valued, with algebraically closed residue field, it is C_1
- **Big Conjecture** (Artin Conjecture) $\mathbb{Q}^{\mathbf{ab}}$ is C_1

3. $K \quad C_r$, $K'/_K$ algebraic $\implies$ K' is C_r

4. $k \quad C_r$, $\mathbf{td}(K/_k) = d \implies K$ is C_{r+d}

   In particular, k algebraically closed, $K/_k$ function field with $\mathbf{td}(K/_k) = d$
   $\implies K$ is C_r ???

Proof of Fact

- $(t_1, \ldots, t_d)$ in $K^\times$ $\qquad a \in K^\times$
- $q = q_{(t_1,\ldots,t_d)} - a x_{2^d+1}^2$ represents 0 where $\mathbf{td}(K/_k) = d$
- $\implies$ $q_{(t_1,\ldots,t_d)}$ represents 0 (Why?)

For the rest

- Let $\mathcal{T} = (t_1, \ldots, t_d)$ be a separable transcendence base
- Let $R = k[t_1, \ldots, t_d]$ $\qquad$ $\mathbf{Spec}\, R \cong \mathbb{A}^d_k$
- S =integral closure of R in K
- $\mathbf{Spec}\, S = X \xrightarrow{\phi} \mathbb{A}^d_k$
- ϕ is étale on an open subset $U \subseteq X$
- Choose $x \in U$ closed point

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{A}^d_k \\ x & \longmapsto & x_o \end{array} \qquad \mathcal{O}_{\mathbb{A}^d_k, x_o} \subseteq \mathcal{O}_{X,x}$$

- $(t_1 - a_1, \ldots, t_d - a_d)$ local parameters of x_o with $(a_1, \ldots, a_d) \in k$
- If follows $(t_1 - a_1, \ldots, t_d - a_d)$ local parameters at x

$$\mathcal{O}_{\mathbb{A}^d,x_o} \hookrightarrow \mathcal{O}_{X,x} \rightsquigarrow \hat{\mathcal{O}}_{\mathbb{A}^d,x_o} \overset{\sim}{=\!=\!=} K[\![t_1-a_1,\ldots t_d-a_d]\!] =\!=\!=\!=\!= \hat{\mathcal{O}}_{X,x}$$

$$\begin{array}{c} K[\![t_1-a_1,\ldots t_d-a_d]\!] \\ \big\downarrow \\ \Lambda :=\!=\!=\!= k(\!(t_a - a_a)\!) \cdots (\!(t_d - a_d)\!) \end{array}$$

- Using the Lemma before: $q_{(t_1-a_1,\ldots,t_d-a_d)}$ does not represent 0 over Λ (Induction on d)