Elementary Equivalence $\iff$ Isomorphism

Introduction Motivation

- “Classifying Question”
  
  $K = \mathbb{C}(x_1, \ldots, x_n) \sim \text{td}(K/\mathbb{C})$

- Elementary Equivalence of Fields
  
  - $\mathcal{L} = +, -, \cdot, 1, 0$ Axioms of fields
  
  - $\mathcal{A} = \{ \text{all sentence in } \mathcal{L} \}$
  
  - $\text{Th}(K) = \{ \phi \in \mathcal{A} \mid K \models \phi \}$
  
  - $\forall \vec{x}_1 \exists \vec{x}_2 \ldots \overline{P}(\vec{x}_1, \ldots) = 0$
• Fact

\[ K \cong L \implies \text{Th}(K) = \text{Th}(L) \]

\[ \iff \]

• Definition (elementary equivalent) \( K \equiv L \iff \text{Th}(K) = \text{Th}(L) \)
Facts

1. $\text{Char}(K)$ is encoded in $\text{Th}(K)$
2. $K$ algebraically closed is encoded in $\text{Th}(K)$
3. $K$ is real closed is encoded in $\text{Th}(K)$
   - $\nu : K^\times \longrightarrow \Gamma_v \quad \Gamma_v$ divisible
   - $k_v \subseteq \mathbb{R}$ is relatively algebraically closed
   - $(K, \nu)$ henselian
4. $K$ $p$-adically closed is encoded in $\text{Th}(K)$
5. Geometric interpretation of $\equiv$
   - Every $\phi \in \mathcal{A} \quad \longleftrightarrow \quad S_\phi \subseteq \mathbb{A}_\mathbb{Z}^N$
   - $K \equiv L \iff$
     - $(\forall \phi \text{ one has } S_\phi(K) \neq \emptyset \iff S_\phi(L) \neq \emptyset)$
7. Relation of $\equiv$ with $\cong$

- Ultrapowers: $K$, $I$ index set, $\mathcal{D}$ ultrafilter on $I$,
  \[ K^* := K^I/\mathcal{D} = K^I/\mathcal{M}_\mathcal{D} \]
- $K \equiv L \iff \exists K^* = K^I/\mathcal{D}$, $L^* = K^J/\mathcal{E}$ such that $K^* \cong L^*$
- **Remark** $K \equiv K^*$
**Theorem** (Classification up to $\equiv$)

1. $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}}_p$ (all $p$) are representatives for all algebraically closed fields
2. $\mathbb{R}^{\text{abs}} := \mathbb{R} \cap \overline{\mathbb{Q}}$ is real closed and all real closed fields are elementary equivalent
3. $\mathbb{Q}^{\text{abs}}_p := \mathbb{Q}_p \cap \overline{\mathbb{Q}}$ is $p$-adically closed and all $p$-adically closed fields are elementary equivalent

**Comment** Generalized $p$-adically closed fields:

$K/\mathbb{Q}_p$ finite, $d := [K : \mathbb{Q}_p]$, $K^{\text{abs}} := K \cap \overline{\mathbb{Q}}$
Consequence if $K \equiv L \implies K^{\text{abs}} \cong L^{\text{abs}}$

Proof

\[ K^* \rightleftharpoons L^* \]

\[ K \leftarrow K^{*, \text{abs}} \rightleftharpoons L^{*, \text{abs}} \leftarrow L \]

\[ K^{\text{abs}} \quad L^{\text{abs}} \leftarrow \]

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The Other Extreme

- prime field $\leftrightarrow$ arithmetic situation
- fields of finite type over $\rightarrow$
- alg. cl. field $\leftrightarrow$ geometric situation

• Comment

- $\mathbb{Q}(t) = \mathcal{K}(\mathbb{P}^1_\mathbb{Q})$
- $\mathbb{Q}(u, v)$
- $\mathcal{F} = (t_1, \ldots, t_d)$ is a transcendence basis (TB) over $k$ if
  $\forall P(x_1, \ldots, x_d) \neq 0 \in k[x_1, \ldots, x_d] \implies P(t_1, \ldots, t_d) \neq 0$
Arithmetic: $K$ Geometric: $K$

\[ k(t_1, \ldots, t_d) \quad \text{finite} \quad k(t_1, \ldots, t_d) \quad \text{finite} \]

\[ k \quad \text{purely transcendental} \quad k \quad \text{purely transcendental} \]

\[ k = \mathbb{F}_p \text{ or } \mathbb{Q} \]
• **Problem** (Elementary Equivalence $\iff$ Isomorphism)
  Describe a set of representatives for the elementary equivalence classes of function fields (arithmetic/geometric situation)

• **Hope** $K, L$ such fields $\iff K \cong L$
(Sabbagh ’85)

- **Theorem A** (Arithmetic Case) Let $K$ and $L$ be arithmetic function fields. Suppose $K \equiv L$ then $\exists$ field embeddings $K \hookrightarrow L$ and $L \hookrightarrow K$. In particular, if $K$ is of general type, then $K \cong L$

- **Theorem B** (Geometric Situation) Let $K$ and $L$ be geometric function fields. Suppose $K \equiv L$ then
  1. $\text{td}(K/k) = \text{td}(L, \lambda)$  \hspace{1em} $k \equiv \lambda$
  2. if $K$ is of general type $K \cong L$ provided that $k \cong \lambda$
Comments

- Durét, Pierce: Geometric case under hypothesis $\text{td}(K/k) = 1$ goes beyond Theorem B: elliptic curves of non-CM type

- $K^* \cong L^*$
  
  $\mathcal{C}_L \longrightarrow \mathcal{C}_K \quad K \leftarrow L$