Manin-Mumford A semi-abelian variety, $\mathbf{Char}(K) = 0$, $X \subseteq A$ subvariety $\implies$ $\overline{X \cap \mathbf{Tor}(A)}$ is a finite union of translates of semi-abelian varities.

- May assume $0 \in X$, $X \cap \mathbf{Tor}(A)$ is Zariski dense in X
- Want that X is an algebraic subgroup
- **Proposition (4.4)** A semi-abelian, $\phi : A \longrightarrow A$ seperable isogeny, $X \subseteq A$, $\phi(X) \subseteq X$, $0 \in X$, X generates A, $\mathbf{Stab}(X) = 0 \implies \phi^n = \mathbf{id}$ for some n

- All over $k = \#$ field
- **Lemma (4.1)** $\exists \sigma \in \mathbf{Gal}\left(\overline{K}/K\right)$, $P(T) \in \mathbb{Z}[T]$ with no roots of unity among its roots such that $\mathbf{Tor}(A) \subseteq \mathbf{Ker}\,(P(\sigma))$ where $P(\sigma) : A(\overline{K}) \longrightarrow A(\overline{K})$
- $(\overline{K}, \sigma)$ is a difference field
- $H := \mathbf{Ker}\,(P(\sigma))$ is a definable subgroup

Reduction

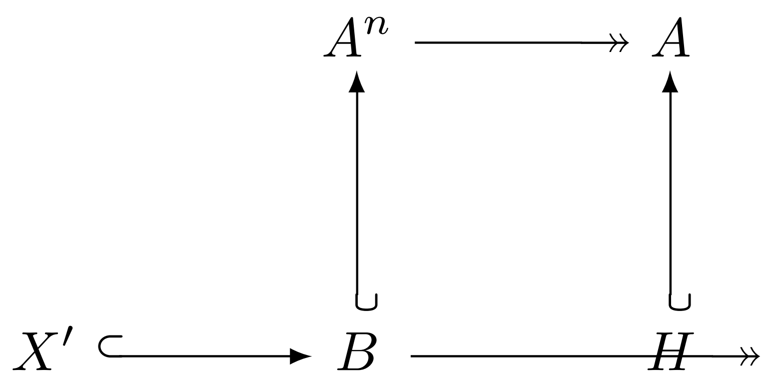
- $H < A$ is a difference definable subgroup of A
- $P(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0$
- Let $A^n = A \times \cdots \times A$
- Let $\pi : A^n \longrightarrow A$ be the projection to the 1^{st} coordinate.
- Let ϕ be the algebraic endomorphism of A^n

$$\phi(x_0, \ldots, x_{n-1}) = (x_0, x_1, \ldots, x_{n-2}, -a_1 x_1 - \cdots - a_{n-1} x_{n-1})$$

- **(4.2)** $P(\phi) = 0$ and so $\forall r\ \phi^r - \mathbf{id}$ is an isogeny of A^n

(4.3a) $\exists$ semi-abelian subvariety $B \subseteq A^n$ defined over k and irreducible subvariety $X' \subseteq B$ (over a finite extension of k) such that:

1. $\pi|_B : B \twoheadrightarrow A$
2. $\phi|_B : B \longrightarrow B$ is an isogeny
3. $\phi^m(X') \subseteq X'$ for some m
4. $\pi(X') \subseteq X'$ is Zariski dense.



Proof

- Let $H_1 = \{\vec{x} \in A^n \quad | \quad \phi(\vec{x}) = \sigma(\vec{x})\}$
- $H_1 = \Big\{\big(x, \sigma(x), \ldots, \sigma^{n-1}(x)\big) \quad | \quad x \in H\Big\}$
- B=connected component of Zariski closure of H_1 in A^n

  (Assume $H_1 \subseteq B$)
- $\pi(H) \cap X$ is Zariski dense in X
- Let $Y = \overline{\pi^{-1}(X) \cap H_1}$ so $\phi(Y) \subseteq Y$
- Let X' = irreducible component of Y such that $\pi(X') \subseteq X$ is Zariski dense
- $\phi^m(X') \subseteq X'$ for some m

Case 1

X' is an algebraic subgroup of $B \implies X$ is an algebraic subgroup of A

Case 2

- Otherwise let $S := \mathbf{Stab}_B(X')$
- Then S is ϕ^m-invariant
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$$X'/S \quad \subseteq \quad B/S$$

↑ positive dimensional with trivial stabalizer

- By **(4.2)** for every r, $\phi^{mr} - 1$ is an isogeny of B/S
- Contradiction to **(4.4)**