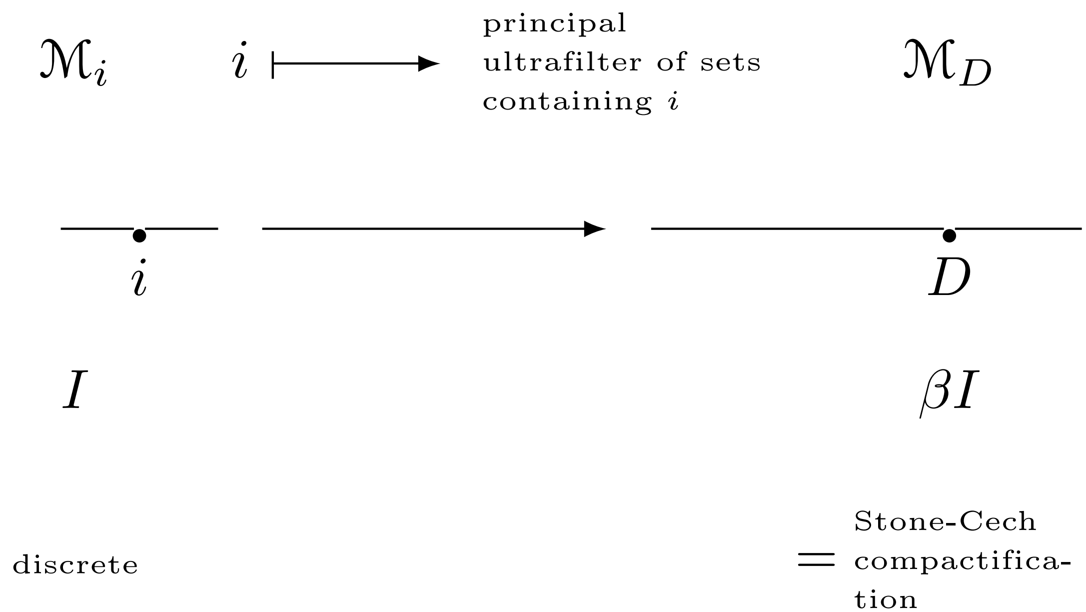
Uniform Elimination for $\mathbb{Q}_p$

$$\Downarrow$$

Uniform Elimination for $\mathbb{F}_p$

Relations to decision problem for finite (or prime) fields

- Family $\mathcal{M}_i \quad i \in I$ of structures



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$$\mathbb{B} = \mathbf{Pow}(I) \quad \begin{array}{lcl} \text{as a boolean algebra} & \longrightarrow & \text{ultrafilters} \\ & & \wr\!\wr\!\wr \\ \text{as a boolean ring} & \longrightarrow & \text{prime (max) ideals} \end{array}$$

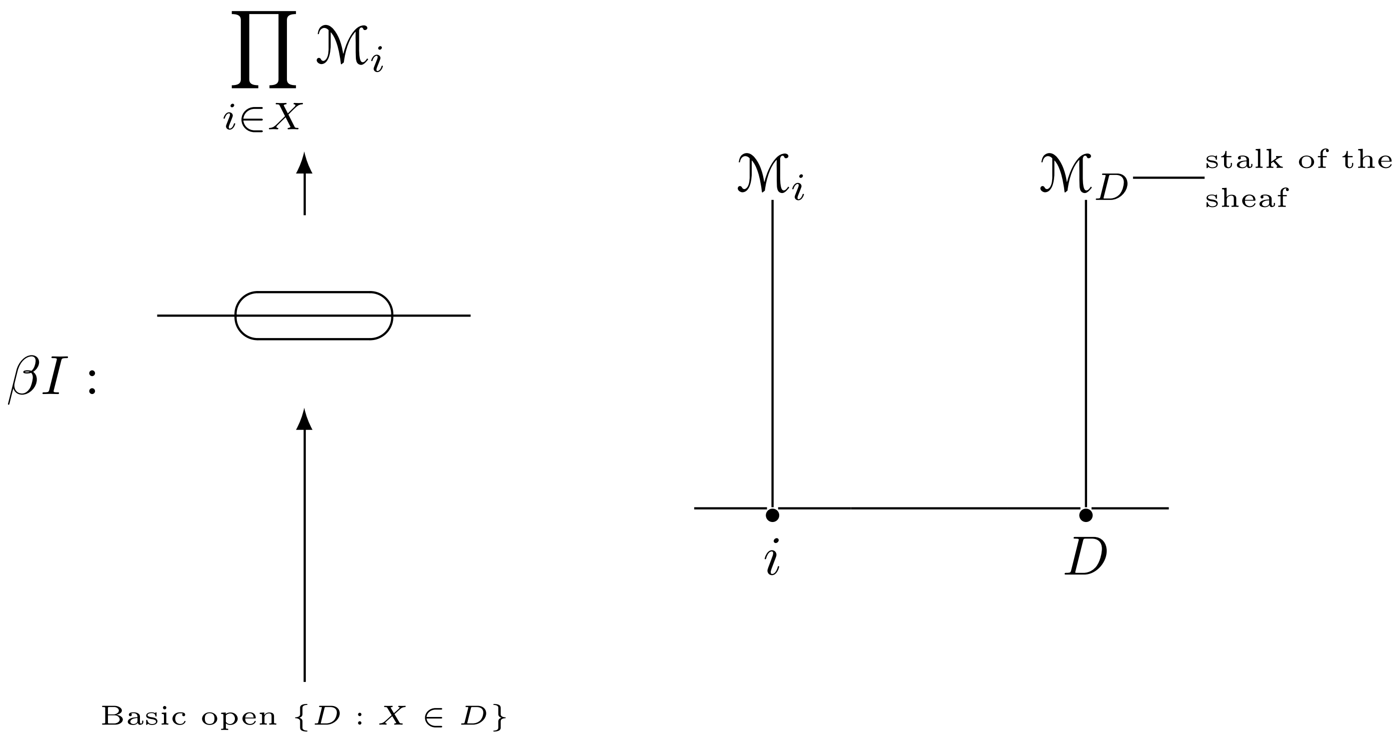
- Topologize $\beta I = \mathbf{Spec}(\text{Boolean Ring})$
- Basic open sets of form

$$\{D : X \notin D\} = \{D : I \setminus X \in D\}$$

Where D is an ultrafilter, $X \subseteq I$, $X \in \mathbb{B}$

- Compact totally disconnected
- D has dually a max ideal in ring

Sheaf


$\prod_{i\in X} \mathcal{M}_i$
βI :
Basic open $\{D : X \in D\}$
$\mathcal{M}_i$
$\mathcal{M}_D$
stalk of the sheaf
i
D

- Elements of $\mathcal{M}_D$
  - germs living on some $\prod_{i \in X} \mathcal{M}_i$ where $X \in D$
  - Write as functions f
- Have "continuity" of satisfaction
- Suppose have elements $i_\alpha \in I$ converging to D

  i.e. if $X \in D$ eventually all $i_\alpha \in X$

- Meaning of $\mathcal{M}_{i_\alpha} \rightsquigarrow \mathcal{M}_D$
- Fix a function $\phi(v_1, \ldots, v_n)$
- Fix $f_1, \ldots, f_n \in \mathcal{M}_D$
- **Theorem**(tos) $\mathcal{M}_D \models \phi(f_1, \ldots, f_n) \iff$

$$\{i \in I : \mathcal{M}_i \models \phi(f_1(i), \ldots, f_n(i))\} \in D$$

- **Meaning** Truth value of $\phi(f_1(i_\alpha), \ldots, f_n(i_\alpha)) \underset{\text{(converges)}}{\longrightarrow\!\!\!\rightsquigarrow}$ truth value of $\phi(f_1(D), \ldots, f_n(D))$

- $$\mathcal{M}_i \; i \in I \qquad \mathcal{M}_D \qquad D \in \beta I$$

  $$\Big|$$

  $$\mathcal{M}_{\mathfrak{p}} \quad \mathfrak{p} \in \mathbf{Spec}(\mathbb{B})$$

- If $\mathcal{M}_i$ are valued fields with residue fields K_i, then $\mathcal{M}_D$ is a valued field with residue field K_D

- I =set of primes

  1) $\mathcal{M}_{\mathfrak{p}} = \mathbb{F}_p$

  2) $\mathcal{M}_{\mathfrak{p}} = \mathbb{Q}_p$

  3) $\mathcal{M}_{\mathfrak{p}} = \mathbb{F}_p^{\mathbf{alg}}$

- What about $\mathcal{M}_D$ in those cases?

- **1)** $K = \mathcal{M}_D$ has **Char** 0

  Properties

  - **Gal**$(K) = \hat{\mathbb{Z}}$
  - K is PAC (pseudo-algebraically closed/regularly closed)

- $$\begin{array}{ccc} K_i^{\mathbf{alg}} & \longrightarrow & (K_i^{\mathbf{alg}})_D \\ \uparrow & & \uparrow \\ K_i & \longrightarrow & K_D \end{array}$$

- Regularly closed: if V is absolutely irreducible variety over K, V has a point in K

- K =totally real algebraic numbers

  $K[i]$ is PAC

2) $\mathbb{Q}_p \dashrightarrow K_D$

- Henselian
- Residue field is pseudo-finite= $(\mathbb{F}_p)_D$
- Residue field is **Char** 0
- Value groups is Presburger group

$\mathbb{F}_p((t)) \dashrightarrow L_D$

- Henselian
- Residue field is same
- Value group is Presburger

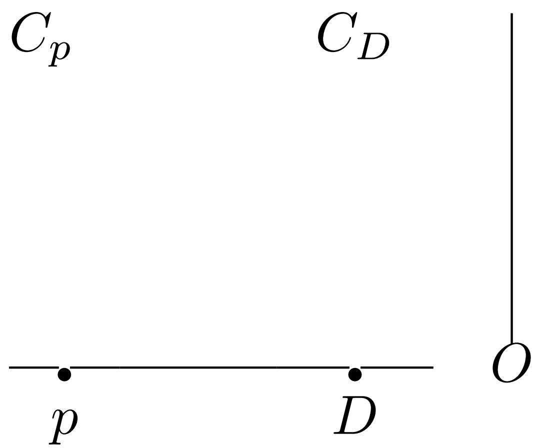
$K_D \equiv L_D$ $\qquad$ $\mathbb{F}_p((t))$ is C_2 thus K_D is C_2

$\implies$ for any instance of C_2, $\mathbb{Q}_p$ satisfies it for $p \geq p_o$

Uniformity

- **Type 1** Within definable sets
- **Type 2** Across p

Basic version of Type 2 Lefschetz principle for algebraically closed fields



$$\mathbb{F}_p^{\mathbf{alg}} \qquad x \overset{\sigma}{\longmapsto} x^p$$

$$\mathbb{F}_p = \mathbf{Fix}(\sigma)$$

$D \rightsquigarrow \sigma_D : \mathbb{C} \longrightarrow \mathbb{C}$

ACFA

Basic Version of Type 1

- $\mathbb{R}$
- $X_{\overline{\lambda}}$ semi-algebraic family of semi-algebraic sets