Axioms for $\mathbb{R}$ $(= K)$

- (1st Set) $\quad \mathbf{Gal}(K) = \mathbb{Z}/_{2\mathbb{Z}}$ $\quad$ 1st order axiom!
- (2nd Set) $\quad$ In language of ordered fields

  SIGN change scheme:

  If $f \in K[x]$ changes sign on $[a, b]$ it has a root on $[a, b]$
- Topology definable: $x > 0 \iff x \neq 0 \wedge P_2(x)$

p-adic Fields (Ax-Kochen, Erson)

- Analogue (2[nd] Set): Hensel's Lemma

  1[st]-order consequence of completions

- Analogies between $\mathbb{R}$ and $\mathbb{Q}_p$

  In $\mathbb{Q}_p$ topology is algebraically definable

$$\mathbb{Q}_p \qquad\qquad \mathbf{v} : \mathbb{Q}_p \longrightarrow \mathbb{Z} \cap \{\infty\}$$

$$\uparrow$$

Presburger

$\mathbb{Z}_p := \{x : \mathbf{v}(x) \geq 0\}$

- $(p \neq 2)$ $x \in \mathbb{Z}_p :$ $(\exists y)\left(y^2 = 1 + px^2\right)$ $\quad \|x\|_p \leq 1$
- $(p = 2)$ $x \in \mathbb{Z}_p :$ $(\exists y)\left(y^2 = 1 + 4px^2\right)$ $\quad \|x\|_2 \leq 1$
- $(p = -1)$ $x \in \mathbb{Z}_p :$ $(\exists y)\left(y^2 = 1 - x^2\right)$ $\quad \|x\| \leq 1 \in \mathbb{R}$

AKE

- Complete set of axioms for certain Henselian fields
- (Generally with 3 sorts)
- Henselian
- Value group a Presburger group
- Complete set of axioms for residue field
- $$\begin{cases} \text{residue field of } \textbf{Char}\, 0 \\ \text{residue field of } \textbf{Char}\, p, \text{ and } \mathbf{v}(p) = 1 \text{ (unramified)} \end{cases}$$
- **Example** $\mathbb{C}((t))$ $\mathbb{R}((t))$

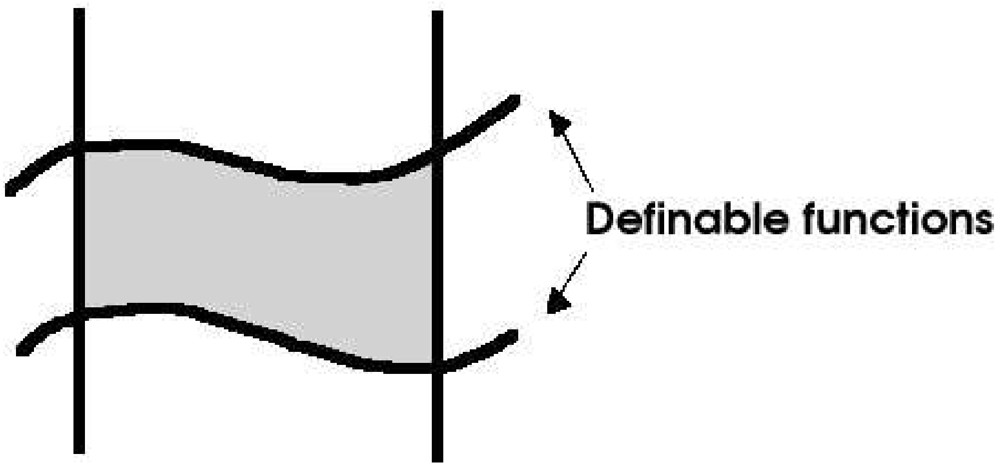
- What about Galois theoretic axioms for $\mathbb{Q}_p$?
- $\mathbf{Gal}(\mathbb{Q}_p)$ Generated topologically by 4 elements

$$\sigma^{-1}\tau\sigma = \tau^p$$

- Conditions on K such that $\mathbf{Gal}(K) \cong \mathbf{Gal}(\mathbb{Q}_p)$ is 1^{st}-order

$$\implies \quad K \equiv \mathbb{Q}_p$$

- Elimination in $\mathbb{Q}_p$ using the P_n
- $\mathbb{Q}_p^\times / _{\mathbb{Q}_p^{\times n}}$ is finite
- **Cells**



- Sides in $\mathbb{Q}_p^n$
  - $f(\overline{x}) \neq 0$ & $P_2(\cdot f(x))$
  - $f(\overline{x}) \neq 0$ & $P_2(-f(x))$

Uniformity in p?

$$\begin{array}{ccc} (\mathbb{Q}_p, P_n) & & (\mathbb{Q}_p, P_{n+1}) \\ | & & | \\ \mathbb{F}_{P_n} & & \mathbb{F}_{P_{n+1}} \end{array}$$

- Limits as $p \to \infty$
- Henselian valued fields
- Value group $\equiv \mathbb{Z}$
- Infinite finite field is residue field $=$

$$\prod \mathbb{F}_p \big/_{\text{non-principal max ideals}}$$