- **Problem**: Associate to formulas geometric objects related to counting points
- $\mathcal{L}$ 1^{st} order language
- T a theory
- $\mathbf{K}_o(T)$ abelian group on 1^{st} order formulas ϕ modulo:

1. - ϕ formula in $\mathfrak{L}$ with free variable $x = (x_1, \ldots x_m)$
   - ϕ' formula in $\mathfrak{L}$ with free variable $x' = (x'{}_1, \ldots x'{}_m)$
   - $[\phi] = [\phi']$ if $\exists \psi \in \mathfrak{L}$ with free variables x and x' such that

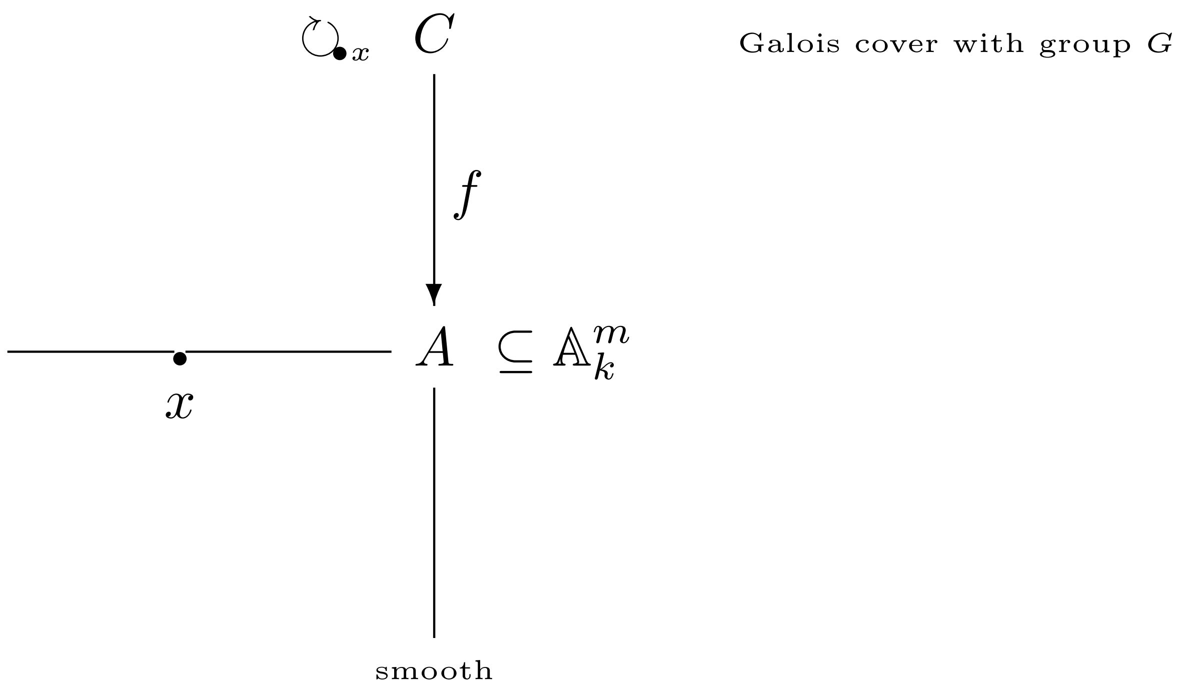
$$T \models [\forall x \, (\phi(X) \longrightarrow \exists ! x' \, (\phi'(x') \wedge \psi(x, x')))]$$

2. $[\phi \vee \phi'] = [\phi] + [\phi'] - [\phi \wedge \phi']$ where ϕ, ϕ' have the same free variables
3. Product: $[\phi(x)][\phi'(x')] = [\phi(x) \wedge \phi'(x')]$ where x and x' are difference sets of free variables

- $\mathbf{PFF}_k$: Theory of pseudo-finite fields$_{/k}$
- $\mathcal{L}$: Language of rings with parameters in k
- $\mathbf{K}_o(\mathbf{PFF}_k) \longrightarrow \mathbf{K}_o^{\mathbf{mot}}(\mathbf{Var}_k) \otimes \mathbb{Q} \qquad \textbf{char } k = 0$

- Consider Chow motives (since isogenous elliptic curves have same # of points in every finite field)
- Kiefe

- Galois stratification (Fried-Sacerdote) (Fried-Jarden)
- Basic Objects:



- $x \in A(k)$, $\exists y \in C(k)$ $\quad f(y) = x \iff$ decomposition group above x is $\{1\}$

- Consider $C \xrightarrow{G} A$ and a family $\mathcal{C}$ of subgroups of G stable under conjugation

- set of points in A with decomposition group in $\mathcal{C}$

- Assume k is a number field, $k_{(\mathfrak{p})}$ finite residue field

- $h(C \xrightarrow{G} A, \mathcal{C}, k_{(\mathfrak{p})}) = \{$ set of points in $k_{(\mathfrak{p})}^n$ with decomposition group in $\mathcal{C}\}$

- $|h(\longrightarrow)|$ expressed in terms of a central function on G built out of $\mathcal{C}$

Chow Motives

- G finite group acting on X a smooth projective variety
- $h(X) = \bigoplus_{\alpha \in \hat{G}} h(X)^{\alpha}$ in $\mathbf{ChMot}_k$

- X smooth projective
- Γ correspondence   i.e. a cycle of dimension $n = \mathbf{dim} X$ in $X \times X$
- **Example**: Γ =graph of a morphism $X \longrightarrow X$

To compose correspondences we need to replace $Z^n(X \times X)$ n-dimensional cycle by $\mathbf{CH}^n(X \times X) \otimes \mathbb{Q}/_\sim$ where $\sim$ is rational equivalence

- Y variety
- $Z,\ Z' \in \mathcal{Z}^d(Y)$ co-dimension d
- If $\exists W \in \mathcal{Z}^{d+1}(Y)$, f meromorphism on W such that $Z - Z' = \mathbf{div}(f)$ then $Z \sim Z'$

A Chow motive$_{/k}$ is (X, p, n) where:

- X smooth projective
- p correspondence on X with $p \circ p = p$
- $n \in \mathbb{Z}$

- To $X \rightsquigarrow h(X) = [X, \mathbf{id}, 0] \in \mathbf{ChMot}_k$

- **Theorem**: (Gillet-Soulé, Guillen-Navarro, Bittner)

  char $k = 0$ $\quad \exists! \mathcal{X}_X : \mathbf{K}_o(\mathbf{Var}_k) \longrightarrow \mathbf{K}_o(\mathbf{ChMot}_k)$ such that $\mathcal{X}_C([X]) = h(x)$ for X smooth projective

Remark $h(\mathbb{P}^1_k) = h(\mathbf{Spec}k) \oplus [\mathbf{Spec}k, \mathbf{id}, 1] \implies$

$\mathcal{X}_c(\mathbb{L}) = \mathbb{L} = [\mathbf{Spec}k, \mathbf{id}, 1]$

- G acts on smooth projective X
- $h(X) = \bigoplus_{\alpha \in \hat{G}} h(X)^{\alpha}$
- $\alpha \rightsquigarrow P_{\alpha}$ a projector in $\mathbb{Q}[G]$
- $P_{\alpha} = \sum \lambda_g g$ where g induces a morphism $f_g : X \longrightarrow X$
- $P_{\alpha}^2 = P_{\alpha}$

- This construction extends to G acting on any X
- $\rightsquigarrow (C \xrightarrow{G} A, \lambda) \rightsquigarrow \mathbf{K}_o(\mathbf{ChMot}_k) \otimes \mathbb{Q}$ where λ is a central function
- In fact belongs to $\mathbf{Im}\,(\mathbf{K}_o(\mathbf{Var}_k) \otimes \mathbb{Q})$ in $\mathbf{K}_o(\mathbf{ChMot}_K) \otimes \mathbb{Q}$
- We denote the image by $\mathbf{K}_o^{\mathbf{mot}}(\mathbf{Var}_K)_{\mathbb{Q}}$
- Finally $\mathcal{X}_C : \mathbf{K}_o(\mathbf{PFF}_k) \longrightarrow \mathbf{K}_o^{\mathbf{mot}}(\mathbf{Var}_k)_{\mathbb{Q}}$
- Well defined – uses Cebotarev
- Finally if ϕ is a formula with coefficients in k, **char** $k = 0$ $\rightsquigarrow$ $\mathcal{X}_C([\phi]) \in \mathbf{K}_o^{\mathbf{mot}}(\mathbf{Var}_k)_{\mathbb{Q}}$
- **Fact**: $\mathbf{Eu}(\mathcal{X}_C([\phi])) \in \mathbb{Z}$

Example

- Take k containing n^{th}-roots of unity
- $\phi_n : \exists y \;\; x = y^n \wedge x \neq 0$
- Then $\mathcal{X}_C([\phi_n]) = \frac{\mathbb{L}-1}{n}$
- $\mathbf{Eu}(\mathcal{X}_c([\phi_n])) = 0$
- $H(\mathcal{X}_C([\phi_n])) = \frac{uv-1}{n}$

- **Define**: $P_r(T) \in \mathbf{K}_o^{\mathbf{mot}}(\mathbf{Var}_k)_{\mathbb{Q}}$
- Assume $X \subseteq \mathbb{A}_k^N$
- There exists formula ϕ_n expressing that points in $\mathcal{L}_n(X)(k)$ lift to $\mathcal{L}(X)(k)$
- $P_r(T) := \sum \mathcal{X}_C([\phi_n])T^n$

Theorem (Denef-Loeser)

1. $P_r(T)$ is rational
2. If k is a number field, $N_{\mathfrak{p}}(P_r(T)) = P_{\mathfrak{p}}(T)$ (Serre's series) for almost all $\mathfrak{p}$

More Generally

- K: number field
- $\mathcal{L}$: Pas-language
- Formula in Pas-language
- f a polynomial (or definable function)
- There exists a motive integral $I_{\mathbf{mot}}$ such that

$$N_{\mathfrak{p}}(I_{\mathbf{mot}}) = \int |f|_{\mathfrak{p}}^{s} |dx|_{\mathfrak{p}}$$

where the integration is taken over the set defined by ϕ (assumed bounded)

General Principle

- Natural p-adic integral are motivic
- **Example** (Habes) Orbital integrals are motivic
- Once can consider motivic integrals with parameters and get a result of Denef-type (Cluckers-Loeser)