Denef gave a proof of Macintyre's theorem using a cell decomposition for which he gave a direct geometric proof

Using cell decomposition we have:

<u>Theorem</u>(Denef)

- $(A_{\lambda,l})_{\substack{\lambda\in\mathbb{Q}_p^m \\ l\in\mathbb{Z}^r}}$ definable family of bounded subsets of $\mathbb{Q}_p^N$
- $\mu_n(A_{\lambda,l})$ is a finite $\mathbb{Q}$-linear combination of functions:

$$\begin{cases} p^{-\alpha(\lambda,l)} \\ \beta(\lambda,l) \end{cases}$$

where α, β are definable $\mathbb{Z}$ valued functions

- Assume no λ i.e. $m = 0$
- Then α, β are Presburger functions
- Deduce

  Corollary

  $\sum_{l \in \mathbb{N}^r} I_l T_1^{l_1} \cdots T_r^{l_r}$ is rational where $I_l = \mu_n(A_l)$
- **Corollary**

  Q and P are rational

Using p-adic Integration to Prove Results over $\mathbb{C}$

- Ax used counting over $\mathbb{F}_q$ to prove results over $\mathbb{C}$
- For $\mathbb{Q}_p$ integration replaces counting

- Take $f \in \mathbb{C}[x_1, \ldots, x_n]$
- Consider $h : Y \longrightarrow X = \mathbb{A}^n_{\mathbb{C}}$
- Assume Y is smooth, h is proper and a birational isomorphism outside of $h^{-1}(0)$
- Assume $h^{-1}(f = 0)$ is a divisor with normal crossings – $h^{-1}(f = 0) = \sum_{i \in J} N_i \, E_i$ where $N_i \in \mathbb{N}$ the E_i are smooth divisors, and E_i's intersect transversally
- h exists by Hironaka

- $I \subseteq J \qquad E_I := \bigcap_{i \in I} E_i \qquad E_I^o := E_I \setminus \bigcup_{j \notin I} E_j$
- Then $Y = \coprod_{I \subset J} E_i^o \qquad E_\emptyset^o = Y \setminus h^{-1}(f = 0)$
- We write $h^* \Omega_X^n = \Omega_Y^n - \sum_{i \in J} (v_i - 1) E_i$ with $v_i = m_i + 1$
- In terms of local coordinates:

$$h^* dx_1 \wedge \cdots \wedge dx_n = u \prod y_i^{v_i - 1} dy_1 \wedge \cdots \wedge dy_n$$

Theorem **(Duenef-Loeser, '92)**

$$Z_{\mathrm{top},t}(s) = \sum_{I \subseteq J} \frac{\mathbf{Eu}(E_I^o)}{\prod_{i \in I}(N_i s + v_i)}$$

is independent of $h : Y \longrightarrow X$.

- $\mathbf{Eu}(W)$ for a complex algebraic variety is given by:

$$\mathbf{Eu}(W) := \sum_{i \geq 0} (-1)^i \mathbf{Rk}\big(\mathbf{H}_c^i\,(W(\mathbb{C}), \mathbb{C})\big)$$

- $\mathbf{Eu}(W) = \mathbf{Eu}(W') + \mathbf{Eu}(W \setminus W')$ for W' closed in W

Sketch of Proof:

- Assume $f \in K[x_1, \ldots, x_n]$ for K a number field
- Consider $\mathfrak{p}$ a prime ideal, $K_\mathfrak{p}$ the completion
- $Z_\mathfrak{p}(s) = \int_{R^m{}_\mathfrak{p}} |f|^s$
- Take $h : Y \longrightarrow X$ defined over K, E_i's also
- **Theorem (Denef)**

  For all most all $\mathfrak{p}$

$$Z_\mathfrak{p}(s) = q^{-n} \sum_{I \subseteq J} \#\left(E_I^o(k_\mathfrak{p})\right) \prod_{i \in I} \frac{(q-1)q^{-(N_i s + v_i)}}{1 - q^{-(N_i s + v_i)}}$$

  where $k_\mathfrak{p}$ is the residue field of $K_\mathfrak{p}$, $q = |k_\mathfrak{p}|$, and $E_I^o(k_\mathfrak{p})$ makes sense for almost all $\mathfrak{p}$
- Idea $q \longmapsto 1$

- Take W a variety over K
- For almost all $\mathfrak{p}$:

$$\begin{aligned} \#(W(k_{\mathfrak{p}})) &= \sum_{i\geq 0}(-1)^i \mathbf{Tr}\Big(\mathbf{Frob}, \mathbf{H}^i_c\left(W, \overline{\mathbb{Q}}_l\right)\Big) \\ &= \sum_{i\geq 0}(-1)^i \left(\sum_{j=1}^{n} \alpha_{i,j}\right) \end{aligned}$$

$$\alpha_{i,j} = \text{eigenevalues of } \mathbf{Frob} \text{ on } \mathbf{H}^i_c\left(W, \overline{\mathbb{Q}}_l\right)$$

- $k_{\mathfrak{p}}^{(e)}$ finite degree e extension of $k_{\mathfrak{p}}$
- $K_{\mathfrak{p}}^{(e)}$ unramified extenstion of degree e of $K_{\mathfrak{p}}$

- $\forall e \geq 1$

$$Z_{\mathfrak{p}}^{(e)}(s) = q^{-ne)} \sum_{I \subseteq J} \# \left(E_I^o(k_{\mathfrak{p}}^{(e)}) \right) \prod_{i \in I} \frac{(q^e - 1)q^{-e(N_i s + v_i)}}{1 - q^{-e(N_i s + v_i)}}$$

- For all most all $\mathfrak{p}$, $\lim_{e \to 0} \# \left(W(k_{\mathfrak{p}}^{(e)}) \right) = \mathbf{Eu}(W)$
- Taking the limit as $e \to 0$ one gets the theorem
- **Remark:** Morally we did integration on $W(\mathbb{F}_q)$ $\quad q \to 1$

Monodromy Conjecture (Igusa)

If s_o is a pole of $\int_{\mathbb{Z}_p^m} |f|^s$ then $\mathbf{exp}(2\pi i s_o)$ is an eigenvalue of the monodromy

- X over $\mathbb{C}$ a Calabi-Yau variety
- X smooth proper of dimension n and $\exists \omega \in \Omega^n_X(X)$ nowhere vanishing
- Mirror symmetry $\implies$ two birationally equivalaent C-Y varieties have the same Hodge numbers

Theorem (Batyrev, '95)

If X and X' are C-Y and birationally equivalent, then

$$b_i(X) = b_i(X') \;\; \forall i \quad \big(b_i = \mathbf{Rk}\big(\mathbf{H}^i\big)\big)$$

Idea

- Given (X, ω), (X', ω')
- Assume X, X' defined over a number field K
- For almost all $\mathfrak{p}$, and all $e \geq 1$

$$\int_{X(K_{\mathfrak{p}}^{(e)})} |w| = \int_{X'(K_{\mathfrak{p}}^{(e)})} |w'|$$

- $\implies$ for almost all $\mathfrak{p}$ and for all $e \geq 1$ that $X(k_{\mathfrak{p}}^{(e)}) = X'(k_{\mathfrak{p}}^{(e)})$
- $\implies$ $b_i(X) = b_i(X')$ $\forall i$ by Weil conjectures

Remark Kontsevich introduced motivic integration and applied it to show that birational C-Y varieties have the same Hodge numbers