ABC ESTIMATE, INTEGRAL POINTS, AND GEOMETRY OF P^n MINUS HYPERPLANES

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Abstract. Let K be a field and $\mathcal{H}$ be a set of hyperplanes in $P^n(K)$. When K is a function field, we show that the following are equivalent. (a) $\mathcal{H}$ is nondegenerate over K. (b) The height of the $(S, \mathcal{H})$-integral points of $P^n(K) - \mathcal{H}$ is bounded. (c) $P^n_K - \mathcal{H}$ is an abc variety. When K is a number field and $\mathcal{H}$ is nondegenerate over K, we establish an explicit bound on the number of $(S, \mathcal{H})$-integral points of $P^n(K) - \mathcal{H}$. Finally, we discuss the geometric properties of holomorphic maps into $P^n(\mathbb{C})$ omitting a set of hyperplanes with moving targets.

0. Introduction

Let F be a number field and $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. Let S be a finite set of valuations of F including all the archimedean valuations. When $\mathcal{H}$ is in general position and the number of hyperplanes in $\mathcal{H}$ is at least $2n+1$, Ru and Wong [RW] proved that the number of the $(S, \mathcal{H})$-integral points is finite; later the author [Wa2] provided an explicit bound on the number. Ru then found a necessary and sufficient condition on $\mathcal{H}$ such that the number of the $(S, \mathcal{H})$-integral points of $P^n(F) - \mathcal{H}$ is finite; he also showed that this is a necessary and sufficient condition of Brody hyperbolicity. However, an explicit bound on the number of the $(S, \mathcal{H})$-integral points was not obtained in [Ru].

Let C be an irreducible nonsingular projective algebraic curve of genus g defined over an algebraically closed field k of characteristic $p \geq 0$. Let K be the function field of C and $\mathcal{H}$ be a set of hyperplanes in $P^n(K)$. Let S be a set consisting of finitely many points of C. When $p = 0$, the author [Wa2] showed that if $\mathcal{H}$ is in general position and the number of hyperplanes in $\mathcal{H}$ is at least $2n+1$ then the height of the $(S, \mathcal{H})$-integral points is bounded and the bound is a linear function of $|S|$. When $p > 0$, the author [Wa3] showed that if $\mathcal{H}$ is in general position and the number of hyperplanes in $\mathcal{H}$ is at least $2n+2$ then under certain condition the height of the $(S,\mathcal{H})$-integral points is bounded and the bound is a linear function of $|S|$.

Recently, motivated by the abc theorem for function fields (cf. [Ma], [BM], [Vol] [Wa1] and [No]), Buium defined abc varieties and proved that any affine open subset of an abelian variety over function fields (of characteristic 0) with trace zero is an abc variety.(cf. [Bu]) The definition of abc varieties is closely related to the (S,D)-integral points of a projective space V deleting a very ample divisor D. It turns out that the previous results on function fields done by the author are all theorems about abc varieties.

In the geometric case, as mentioned before that Ru gave a necessary and sufficient condition for $P^n(\mathbb{C})-\mathcal{H}$ to be Brody hyperbolic. A more general question to consider is when the hyperplanes in $\mathcal{H}$ are moving, i.e. the coefficients of the linear forms corresponding to $\mathcal{H}$ are holomorphic functions. In [Wa4], the author applied the method in [Wa2] and obtained a generalization of the Picard's theorem with moving targets.

In this paper, we will improve the number field result in [Ru] by giving an explicit bound on the number of the $(S,\mathcal{H})$-integral points. In the function field case of zero characteristic, we will show that the condition on $\mathcal{H}$ given in [Ru] is also necessary and sufficient for the height of the $(S,\mathcal{H})$-integral points to be bounded; and is also a necessary and sufficient condition for $P^n_K-\mathcal{H}$ to be an abc variety. Therefore, we will prove that $P^n_K-\mathcal{H}$ is an abc variety if and only if the height of the $(S,\mathcal{H})$-integral points of $P^n(K)-\mathcal{H}$ is bounded. Finally, in the geometric case we deal with the situation when the coefficients of the linear forms corresponding to the hyperplanes in $\mathcal{H}$ are holomorphic functions.

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1. abc varieties and (S,D)-integral points

In this section we will restrict ourselves to function fields. However, the definition of abc varieties and (S,D)-integral points can be easily adapted to number fields.

Let C be an irreducible nonsingular projective algebraic curve of genus g defined over an algebraically closed field k of characteristic $p\geq 0$. Let K be the function field of C. Given a point $P\in C$, we denote by v_P the normalized valuation associated to P. For elements $f_0,...,f_n$ of K, not all zeros, we define the height as

$$h(f_0,...,f_n):=\sum_{P\in C}-\min\{v_P(f_0),...,v_P(f_n)\}.$$

For an element f of K, we define the height as

$$h(f) := \sum_{P\in C} -\min\{0,\ v_P(f)\}.$$

We now recall the definitions of (S,D)-integral points (cf. [Voj]) and abc varieties (cf. [Bu]). Let V be a projective variety defined over K. Let D be a very ample effective divisor on V and let $1=\phi_0,\phi_1,...,\phi_N$ be a basis of the vector space:

$$\mathcal{L}(D)=\{f|f \text{ is a rational function on } V \text{ such that } f=0 \text{ or } (f)\geq -D\}.$$

Then $\phi=(\phi_0,....,\phi_N)$ defines a morphism from V to P^N; and $\tau\rightarrow(\phi_1(\tau),...,\phi_N(\tau))$ is an embedding of $V(K)-D$ into K^N.

Definition. A point τ of $V(K)-D$ is said to be an (S,D)-integral point if $v_P(\phi_i(\tau))\geq 0$, $1\leq i\leq N$, for every $P\notin S$.

Following [Bu] we may define height and conductor as following:

$$\begin{aligned}
&h_\phi(\tau)=h(\phi(\tau))=h(\phi_0(\tau),...,\phi_N(\tau)),\\
&\mathrm{Cond}_\phi(\tau)=\{P\in C:\min\{v_P(\phi_1(\tau)),\dots,v_P(\phi_N(\tau))\}<0\},\\
&\mathrm{cond}_\phi(\tau)=|\mathrm{Cond}_\phi(\tau)|.
\end{aligned}$$

Definition. We say that $V-D$ satisfies the abc estimate over K if

$$h_\phi(\tau)\ll \mathrm{cond}_\phi(\tau)+O(1),\quad \text{for every } \tau\in V(K)-D,$$

where " $\ll$ " means the inequality holds up to multiplication with a positive constant.

Remark. This definition does not depend on the choice of the embedding.(cf. [Bu]) When there is no confusion, we will omit the subscript ϕ.

Definition. V_K-D is an abc variety if it satisfies the abc estimate over every finite extension L of K.

In this paper, we only consider the case when $V=P^n_K$ and D is a set of hyperplanes in $P^n(K)$. Let $\mathcal{H}$ be a set of q distinct hyperplanes in $P^n(K)$ and let L_i, $1\leq i\leq q$, be the linear forms corresponding to $\mathcal{H}$. Then we can fix an embedding from $P^n(K)-\mathcal{H}$ to K^N in the following form

$$(x_0^q/\prod_{i=1}^q L_i(\mathbf{x}),\dots,x_n^q/\prod_{i=1}^q L_i(\mathbf{x}),\dots),$$

where each coordinate of the embedding is in the form $x_0^{i_0}x_1^{i_1}\dots x_n^{i_n}/\prod_{i=1}^{q} L_i(\mathbf{x})$ with $\sum_{j=0}^{n} i_j = q$. Let

$$\phi = (1, x_0^q/\prod_{i=1}^{q} L_i(\mathbf{x}), \dots, x_n^q/\prod_{i=1}^{q} L_i(\mathbf{x}), \dots).$$

Suppose that η is a point in $P^n(K)$ and is represented by $(f_0, ..., f_n)$. Then from the definition of height

$$h(f_0^q, ..., f_n^q) \le h(\prod_{i=1}^{q} L_i(f_0, ..., f_n), f_0^q, ..., f_n^q, ...) = h(\phi(\eta)).$$

On the other hand, if $L_i = \sum_{j=0}^{n} a_{ij}x_j$ then $v_P(L_i(f_0, ..., f_n)) \le \min\{v_P(f_0), ..., v_P(f_n)\}+$ $\min\{v_P(a_{i0}), ..., v_P(a_{in})\}$. Therefore

$$h(\phi(\eta)) = h(\prod_{i=1}^{q} L_i(f_0, ..., f_n), f_0^q, ..., f_n^q, ...) \le qh(f_0, ..., f_n) + \sum_{i=1}^{q} h(a_{i0}, ..., a_{in}).$$

Together we have

$$qh(f_0, ..., f_n) \le h(\phi(\eta)) \le qh(f_0, ..., f_n) + O(1).$$

Proposition 1. *Let $\mathcal{H}$ be a set of q distinct hyperplanes in $P^n(K)$. If $P_K^n - \mathcal{H}$ is an abc variety, then the height of the $(S, \mathcal{H})$-integral points of $P^n(K) - \mathcal{H}$ is bounded linearly in $|S|$.*

Proof. Let τ be an $(S, \mathcal{H})$-integral point of $P^n(K) - \mathcal{H}$ and be represented by $(f_0, ..., f_n)$. Then from the definitions of Cond_ϕ and $(S, \mathcal{H})$-integral points we have

$$\mathrm{cond}_\phi(\tau) \le |S|.$$

$P_K^n - \mathcal{H}$ is an abc variety, hence

$$h_\phi(\tau) \ll \mathrm{cond}_\phi(\tau) + O(1).$$

Since $qh(f_0, ..., f_n) \le h_\phi(\tau)$, we have

$$h(f_0, ..., f_n) \ll |S| + O(1).$$

Remark. This proposition is true for number fields.

2. Further results in function fields

Let F be a number field and let $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. Ru gave a necessary and sufficient condition on $\mathcal{H}$ such that $P^n(F) - \mathcal{H}$ has only finitely many $(S, \mathcal{H})$-integral points. We will show in this section that for a function field K this is a necessary and sufficient condition for $P^n_K - \mathcal{H}$ to be an abc variety; and also a necessary and sufficient condition such that the height of the $(S, \mathcal{H})$-integral points of $P^n(K) - \mathcal{H}$ to be bounded.

We recall some definitions and results from [Ru].

Notation. Let $\mathcal{L}$ be a set of linear forms in $n+1$ variables which are pairwise linearly independent. We denote by $(\mathcal{L})_F$ the vector space generated by the linear forms in $\mathcal{L}$ over F.

Definition. Let F be a field and $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. We let $\mathcal{L}$ be the set of linear forms corresponding to $\mathcal{H}$. (We note here that all linear forms in $\mathcal{L}$ are pairwise linearly independent over F.) $\mathcal{H}$ is said to be nondegenerate over F if $\dim(\mathcal{L})_F = n+1$ and for each proper nonempty subset $\mathcal{L}_1$ of $\mathcal{L}$

$$(\mathcal{L}_1)_F \cap (\mathcal{L} - \mathcal{L}_1)_F \cap \mathcal{L} \neq \emptyset.$$

Remark. If $\mathcal{H}$ is in general position and the number of hyperplanes of $\mathcal{H}$ is no less than $2n+1$, then $\mathcal{H}$ is nondegenerate over F.

Definition. Let F be a field and $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. Let V be a subspace of $P^n(F)$. V is called $\mathcal{H}$-admissible if V is not contained in any hyperplane in $\mathcal{H}$.

Proposition(Ru). *Let $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. Then $\mathcal{H}$ is nondegeneate over F if and only if for every $\mathcal{H}$-admissible subspace V of $P^n(F)$ of projective dimension greater than or equal to one, $\mathcal{H} \cap V$ contains at least three distinct hyperplanes which are linearly dependent over F.*

We will need the following version of the abc theorem [Br-Ma] for function fields of characteristic 0.

Theorem (Brownawell-Masser). *Let the characteristic of K be zero. If $f_0, ..., f_n$ are S-units and $f_0 + \cdots + f_n = 1$, then either some proper subsum of $f_0 + \cdots + f_n$ vanishes or*

$$h(f_0, ..., f_n) \leq \frac{n(n+1)}{2} \max\{0,\ 2g - 2 + |S|\}. \tag{1}$$

We also need the following version of abc theorem [Wa1] for function fields of positive characteristic.

Theorem (Wang). *Let the characteristic of K be a positive integer p. Suppose that $f_0, ..., f_{n+1}$ are S-units of K. If $f_0 + \cdots + f_n = f_{n+1}$ and $f_0, ..., f_n$ are linearly independent over K^{p^m} for some positive integer m, then*

$$h(f_0, ..., f_n) \leq \frac{n(n+1)}{2} p^{m-1} \max\{0, 2g-2+|S|\}. \tag{2}$$

The main results in this section are the following.

Theorem 1. *Let K be the function field of a nonsingular projective algebraic curve C which is defined over an algebraically closed field k with zero characteristic. Let S be a set consisting of finitely many points of C such that there exist nonconstant S-units. Let $\mathcal{H}$ be a set of hyperplanes in $P^n(K)$. Then the following are equivalent*

(a) *$\mathcal{H}$ is nondegenerate over K.*
(b) *$P^n_K - \mathcal{H}$ is an abc variety.*
(c) *The height of the $(S, \mathcal{H})$-integral points of $P^n(K) - \mathcal{H}$ is bounded linearly in $|S|$.*
(d) *The height of the $(S, \mathcal{H})$-integral points of $P^n(K) - \mathcal{H}$ is bounded.*

Theorem 2. *Let K be the function field of a nonsingular projective algebraic curve C which is defined over an algebraically closed field k with characteristic $p > 0$. Let S be a set consisting of finitely many points of C. Let $L_i = X_i$, $0 \leq i \leq n$, and $L_{n+1+i} = \sum_{j=0}^{n} a_{ij} X_j$, $0 \leq i \leq n$, where a_{ij} are elements of K. Let $\mathcal{H}$ be the set of 2n+2 hyperplanes defined by L_i, $0 \leq i \leq 2n+1$. Let S_n be the permutation group of $\{0, 1, 2, .., n\}$. If $\mathcal{H}$ are in general position, i.e. any $n+1$ linear forms corresponding to $\mathcal{H}$ are linearly independent, and the set $\{\Pi_{i=0}^{n} a_{i\sigma(i)} | \ \sigma \in S_n\}$ is linearly independent over k, then $P^n_K - \mathcal{H}$ is an abc variety.*

Proof of Theorem 1. We first show that (a) implies (b). Let $\mathcal{L} = \{L_1, ..., L_q\}$ be the set of linear forms corresponding to $\mathcal{H}$. Let τ be a point of $P^n(K) - \mathcal{H}$ and be represented by $(f_0, ..., f_n)$. Denote by $l_i = L_i(f_0, ..., f_n)$. Let $S_{\mathcal{H}}$ be a set consisting of finitely many points of C such that every coefficient of each linear form L_i has no zero or pole outside $S_{\mathcal{H}}$. Therefore, $v_P(l_i) \geq \min\{v_P(f_0), ..., v_P(f_n)\}$ for $P \notin S_{\mathcal{H}}$. On the other hand, from the definition of $\text{Cond}(\tau)$, we have $q v_P(f_j) \geq \sum_{i=1}^{q} v_P(l_i)$, $0 \leq j \leq n$, for every $P \notin \text{Cond}(\tau)$. Therefore

$$v_P(l_i) = \min\{v_P(f_0), ..., v_P(f_n)\}, \quad \text{for } P \notin \text{Cond}(\tau) \cup S_{\mathcal{H}}, \ 1 \leq i \leq q. \tag{1}$$

Suppose that the set $\{L_{i_1}, ..., L_{i_m}\}$ is linearly dependent over K and every proper subset of $\{L_{i_1}, ..., L_{i_m}\}$ is linearly independent over K. Then we have a linear equation

$$a_{i_1} L_{i_1}(X) + \cdots + a_{i_m} L_{i_m}(X) \equiv 0, \tag{2}$$

where $a_{i_j} \in K^{\times}$. We call equation (2) a minimal relation. Since elements of $\mathcal{L}$ are linear forms in $n+1$ variables and are pairwise linearly independent over K, we have $3 \le m \le n+2$. It is clear that up to a nonzero factor in K there are only finitely many such minimal relations for the set $\mathcal{L}$. Throughout the proof we will fix a finite set of minimal relations representing all minimal relations for $\mathcal{L}$ up to a nonzero factor in K. Without loss of generality, we can enlarge the size of $S_{\mathcal{H}}$. Therefore, we will assume that every coefficient of the minimal relations in this finite set has no zero or pole outside of $S_{\mathcal{H}}$. Let $S_\tau = \mathrm{Cond}(\tau) \cup S_{\mathcal{H}}$. We now consider equation (2). After rearranging the index, we may assume that

$$a_1 L_1(X) + \cdots + a_m L_m(X) \equiv 0, \tag{3}$$

where a_i, $1 \le i \le m$, is an S_τ-unit. Then we have the following equation.

$$a_1 l_1 + \cdots + a_m l_m = 0.$$

After rearranging the index we may assume that $a_1 l_1 + \cdots + a_u l_u = 0$ with $u \le m$ and no proper subsum of $a_1 l_1 + \cdots + a_u l_u$ vanishes. Therefore

$$\frac{a_2 l_2}{a_1 l_1} + \cdots + \frac{a_u l_u}{a_1 l_1} = -1. \tag{4}$$

By (1), $\frac{l_i}{l_1}$, $1 \le i \le q$, is an S_τ unit. Hence by the theorem of Brownawell and Masser we have

$$h(\frac{a_i l_i}{a_1 l_1}) \le \frac{u(u-1)}{2} \max\{0, 2g-2+|S_\tau|\}, \quad 1 \le i \le u. \tag{5}$$

From the definition of height,

$$h(\frac{l_i}{l_1}) \le h(\frac{a_i l_i}{a_1 l_1}) + h(\frac{a_i}{a_1}). \tag{6}$$

The coefficients $a_i's$ in the representing set of minimal relations only depend on $\mathcal{L}$ and the number is finite. Therefore inequalities (5) and (6) imply

$$h(\frac{l_i}{l_1}) < \frac{n(n+1)}{2}|S_\tau| + O_1, \tag{7}$$

where $1 \le i \le u$ and O_1 only depends on $\mathcal{L}$ and can be determined effectively. From now O_i always represents a constant which only depends on $\mathcal{L}$ and can be determined effectively. If the dimension of the vector space spanned by $L_1, ..., L_u$ over K is $n+1$, then after a linear transformation one can show that (cf. [Wa2])

$$h(f_0, ..., f_n) \le \frac{n^2(n+1)}{2}|S_\tau| + O_2. \tag{8}$$

If the dimension of the vector space spanned by $L_1, ..., L_u$ over K is less than $n+1$, then the set $\{L_1, ..., L_u\}$ is a proper subset of $\mathcal{L}$. Since $\mathcal{H}$ is nondegenerate,

$$(L_1, ..., L_u)_K \cap (L_{u+1}, ..., L_q)_K \cap \mathcal{L} \neq \emptyset. \tag{9}$$

Suppose that $L_i \in (L_1, ..., L_u)_K \cap (L_{u+1}, ..., L_q)_K$. If $1 \leq i \leq u$, then after rearranging the linear forms we have $L_i = a_{u+1}L_{u+1} + a_{u+2}L_{u+2} \cdots + a_w L_w$, where $a_j \neq 0$ and is assumed previously to be an S_τ-unit. Similarly, after rearranging the index, we have an equation

$$l_i = a_{u+1}l_{u+1} + a_{u+2}l_{u+2} \cdots + a_\nu l_\nu, \quad \nu \leq w,$$

where no proper subsum of the equation vanishes. Therefore we have

$$h(\frac{l_{u+1}}{l_i}) \leq \frac{n(n+1)}{2}|S_\tau| + O_3. \tag{10}$$

Hence,

$$\begin{aligned} h(\frac{l_{u+1}}{l_1}) &\leq h(\frac{l_{u+1}}{l_i}) + h(\frac{l_i}{l_1}) \\ &\leq n(n+1)|S_\tau| + O_4. \end{aligned} \tag{11}$$

If $i \geq u+1$, after rearranging the index we may assume that $i = u+1$. Then we have $L_{u+1} = a_{i_1}L_{i_1} + \cdots + a_{i_w}L_{i_w}$, where $\{i_1, ..., i_w\}$ is an index subset of $\{1, ..., u\}$ and a_{i_j} is an S_τ-unit. Similarly, we have

$$h(\frac{l_{u+1}}{l_{i_j}}) \leq \frac{n(n+1)}{2}|S_\tau| + O_5.$$

Therefore

$$\begin{aligned} h(\frac{l_{u+1}}{l_1}) &\leq h(\frac{l_{u+1}}{l_{i_j}}) + h(\frac{l_{i_j}}{l_1}) \\ &\leq n(n+1)|S_\tau| + O_6. \end{aligned} \tag{12}$$

Hence, we have showed that $h(\frac{l_i}{l_1}) \leq O_7|S_\tau| + O(1)$ for $1 \leq i \leq u+1$. If $(L_1, ..., L_{u+1}) = (\mathcal{L})$, then we are done. Otherwise, we can repeat the same argument. Since $\dim(\mathcal{L})_K = n+1$, after repeating the argument finitely many times we can find linear forms $L_1, ..., L_w$ such that $\dim(L_1, ..., L_w)_K = n+1$ and $h(\frac{l_i}{l_1}) \leq O_8|S_\tau| + O_9$, $1 \leq i \leq w$. Therefore, after a linear transformation (cf. [Wa2])

$$h(f_0, ..., f_n) \leq O_{10}|S_\tau| + O_{11} \ll \mathrm{cond}(\tau) + O(1). \tag{13}$$

It is clear from the proof that the abc estimate holds for every finite extension of K. This shows that $P^n_K - \mathcal{H}$ is an abc variety.

It follows from Proposition 1 that (b) implies (c). (c) implies (d) trivially. It remains to show that (d) implies (a). We follow the arguments in [Ru]. Assume that $\mathcal{H}$ is not nondegenerate over K. Then there exists an $\mathcal{H}$-admissible subspace V of $P^n(K)$ of projective dimension greater than or equal to 1 such that $\mathcal{H}\cap V$ does not contain at least three distinct hyperplanes which are linearly dependent over K. After linear changing of basis we may assume that $V = P^m(K)$, $m \le n$. Then $\mathcal{H}\cap V$ contains exactly q distinct hyperplanes which are linearly independent over K and $q \le m+1$. Without loss of generality we may assume that $V = P^n(K)$ and $\mathcal{H}$ contains exactly $q \le n+1$ distinct coordinate hyperplanes. Let f be a nonconstant S-unit. Then the point in $P^n(K)-\mathcal{H}$ represented by $(1, f^{|S|^r}, ..., f^{|S|^r})$ is an $(S,\mathcal{H})$-integral point and

$$h(1, f^{|S|^r}, ..., f^{|S|^r}) = |S|^r h(f) \ge |S|^r. \tag{14}$$

This shows that (d) implies (a).

The proof in [Wa3] can be easily modified to show Theorem 2. For convenience of readers, we give an outline of the proof.

Proof of Theorem 2. Since $k = \cap_{i=0}^{\infty} K^{p^i}$ [GV], if the set $\{\Pi_{i=0}^n a_{i\sigma(i)} |\ \sigma \in S_n\}$ is linearly independent over k, then there exists a positive integer m which depends only on a_{ij} such that the set $\{\Pi_{i=0}^n a_{i\sigma(i)} |\ \sigma \in S_n\}$ is linearly independent over K^{p^m}.

Let $(f_0, ..., f_n)$ represent a point τ in $P^n(K) - \mathcal{H}$. Denote by $l_i = L_i(f_0, ..., f_n)$. Let $S_{\mathcal{H}}$ be a finite subset of C such that every coefficient of each linear form has no zero or pole outside $S_{\mathcal{H}}$. Therefore (1) gives

$$v_P(l_i) = \min\{v_P(f_0), ..., v_P(f_n)\} \quad \text{for } P \notin \mathrm{Cond}(\tau)\cup S_{\mathcal{H}},\ 1 \le i \le q. \tag{15}$$

Since $f_i = l_i \ne 0$, $0 \le i \le n$, we may assume that $f_n = 1$. Let $S_\tau = \mathrm{Cond}(\tau)\cup S_{\mathcal{H}}$. (15) then implies that l_i is an S_τ-unit for $0 \le i \le 2n+1$. Since $L_0, ..., L_{2n+1}$ are in general position, $a_{ij} \ne 0$. Without loss of generality, we let $a_{in} = 1$ for every $n+1 \le i \le 2n+1$. Then we have the following S_τ-unit equations

$$a_{i0}f_0 + a_{i1}f_1 + \cdots + a_{i,n-1}f_{n-1} + 1 = l_{n+1+i}, \quad 0 \le i \le n.$$

If $a_{\beta 0}f_0, a_{\beta 1}f_1, ..., a_{\beta,n-1}f_{n-1}$, and 1 are linearly independent over K^{p^m} for some $0 \le \beta \le n$, then by the theorem of Wang, for $0 \le j \le n-1$,

$$\begin{aligned} h(a_{\beta j}f_j) &\le h(a_{\beta 0}f_0, a_{\beta 1}f_1, ..., a_{\beta,n-1}f_{n-1}, 1) \\ &\le \frac{n(n+1)}{2} p^{m-1} \max\{0,\ 2g-2+|S_\tau|\}. \end{aligned}$$

From the definition of height we have the following abc estimate

$$\begin{aligned} h(f_0,...,f_n) &\le h(a_{i0}f_0,a_{i1}f_1,...,a_{i,n-1}f_{n-1},a_{i,n}f_n)+h(\frac{1}{a_{i0}},...,\frac{1}{a_{in}}) \\ &\le \frac{n(n+1)}{2}p^{m-1}\max\{0,\ 2g-2+|S_\tau|\}+\sum_{0\le i,j\le n} h(a_{ij}) \\ &\le \frac{n(n+1)}{2}p^{m-1}\max\{0,\ 2g-2+\mathrm{cond}(\tau)+S_{\mathcal{H}}\}+\sum_{0\le i,j\le n} h(a_{ij}). \end{aligned}$$

Therefore, we only need to consider the case where each set $\{a_{i0}f_0,...,a_{i,n}f_n\}$, $0\le i\le n$, is linearly dependent over K^{p^m}. The next lemma shows that this is impossible if $\{\Pi_{i=0}^n a_{i\sigma(i)}|\ \sigma\in S_n\}$ is linearly independent over K^{p^m}. The theorem is then proved. □

Lemma. *Let f_i and a_{ij}, $0\le i,j\le n$, be non-zero elements of a field E. If each set $\{a_{i0}f_0,a_{i1}f_1,...,a_{i,n}f_n\}$, $0\le i\le n$, is linearly dependent over a subfield F of E, then the set $\{\Pi_{i=0}^n a_{i\sigma(i)}|\ \sigma\in S_n\}$ is linearly dependent over F.*

Proof. See [Wa3].

3. The explicit bound for number fields

The proof of Theorem 1 can be adapted to the number field case directly. However, the S-unit theorem for number fields only provides an explicit bound on the number of S-unit solutions. Therefore, our method can provide explicit bound on the number of $(S,\mathcal{H})$-integral points, but can not say anything about the abc estimate. Let F be a number field of degree d. Denote by M_F as the set of valuations of F and by M_∞ as the set of archimedean valuations of F. We first recall the S-unit theorem by Schlickewei [Sc]:

Theorem (Schlickewei). *Let $a_1,...,a_n$ be nonzero elements of F. Suppose that S is a finite subset of M_F of cardinality s, containing M_∞. Then the equation*

$$a_1x_1+\cdots+a_nx_n=1 \tag{16}$$

has no more than

$$(4sd!)2^{36nd!s^6} \tag{17}$$

solutions in S-units $x_1,...,x_n$ such that no proper subsum $a_{i_1}x_{i_1}+\cdots+a_{i_m}x_{i_m}$ vanishes.

When $\mathcal{H}$ is nondegenerate to provide an explicit bound on the number of $(S,\mathcal{H})$-integral points we can apply the same method in the proof of Theorem 1. Since it is completely parallel to the function field case, we will only reproduce the parts which need the S-unit theorem.

Following the first part of the proof of Theorem 1, we apply the S-unit theorem to equation (4) in the number field case. Then we showed that the number of the S-unit solutions $(\frac{l_2}{l_1}, ..., \frac{l_u}{l_1})$ is no more than $(4sd!)2^{36ud!s^6}$. Therefore the number of $\frac{l_i}{l_1}$, $2 \leq i \leq u$, which satisfies (4), is no more than

$$(4sd!)2^{36ud!s^6}.$$

Again if the dimension of the vector space spanned by $L_1, ..., L_u$ over F is $n+1$, then without loss of generality we may assume that $L_1, ..., L_{n+1}$ are linearly independent over F. Therefore the number of $(S, \mathcal{H})$-integral points is equal to the number of $(1, \frac{l_2}{l_1}, ..., \frac{l_{n+1}}{l_1})$ which is bounded by

$$(4sd!)^n 2^{36n(n+1)d!s^6}. \tag{18}$$

If the dimension of the vector space spanned by $L_1, ..., L_u$ over F is less than $n+1$, then we can repeat the method in Theorem 1 and establish the same bound (18) for the number of $(S, \mathcal{H})$-integral points. Therefore, together with Ru's result (cf. [Ru]) we have the following

Theorem 3. *Let F be a number field of degree d. Suppose that S is a finite subset of M_F of cardinality s, containing M_∞. Let $\mathcal{H}$ be a set of hyperplanes in $P^n(F)$. $\mathcal{H}$ is nondegenerate if and only if the number of $(S, \mathcal{H})$-integral points of $P^n(F) - \mathcal{H}$ is finite. Furthermore, the number of $(S, \mathcal{H})$-integral points of $P^n(F) - \mathcal{H}$ is bounded by*

$$(4sd!)^n 2^{36n(n+1)d!s^6}.$$

4. A generalization of the Picard's theorem

A complex space M is called Brody hyperbolic if every holomorphic curve $f : \mathbb{C} \to M$ is constant. Ru proved the following (cf. [Ru]).

Theorem (Ru). *Let $\mathcal{H}$ be a set of hyperplanes in $P^n(\mathbb{C})$. Then $P^n(\mathbb{C}) - \mathcal{H}$ is Brody hyperbolic if and only if $\mathcal{H}$ is nondegenerate over $\mathbb{C}$.*

In [Wa4] we extended the classical Picard's theorem to the case where the coefficients of the linear forms corresponding to $\mathcal{H}$ are holomorphic functions. In this section we will improve the results by adapting the proof of Theorem 1.

First, we explain our notation and terminologies. Let $L_i(z)(X) = \sum_{j=0}^{n} g_{ij}(z)X_j$, $1 \leq i \leq q$, $z \in \mathbb{C}$, where g_{ij} are holomorphic functions and for each i, $g_{i0}, ..., g_{in}$

has no common zeroes . Denote by $H_i(z) = \{(x_0,...,x_n) \in P^n(\mathbb{C})|\ L_i(z)(x_0,...,x_n) = 0\}$ as the corresponding moving hyperplane of $L_i(z)$, $1 \le i \le q$, $z \in \mathbb{C}$, and let $\mathcal{H}(z) = \{H_1(z),...,H_q(z)\}$. Let f_0,...,f_n be holomorphic functions without common zeroes. We say that a holomorphic map f represented by $(f_0,...,f_n)$ is a holomorphic map omitting $\mathcal{H}(z)$ if $L_i(z)(f_0(z),...,f_n(z)) \neq 0$ for each $z \in \mathbb{C}$ and $i = 1,...,q$.

Denote by Hol($\mathbb{C}$) as the ring consisting of all holomorphic functions on $\mathbb{C}$, and Mero($\mathbb{C}$) as the field consisting of all meromorphic functions on $\mathbb{C}$. One can identify H_i as a hyperplane in $P^n(\text{Hol}(\mathbb{C}))$. Then $\mathcal{L} = \{L_1,...,L_q\}$ can be identified as a set of linear forms with holomorphic functions as coefficients. Suppose that the set $\{L_{i_1},...,L_{i_m}\}$ is linearly dependent over Mero($\mathbb{C}$) and any proper subset of $\{L_{i_1},...,L_{i_m}\}$ is linearly independent over Mero($\mathbb{C}$). Then we have a minimal relation

$$a_{i_1}L_{i_1}(X) + \cdots + a_{i_m}L_{i_m}(X) \equiv 0, \tag{19}$$

where a_{i_j} is a nonzero holomorphic function and $a_{i_1},\ldots,a_{i_m}$ has no common zeros.

Definition. $\mathcal{H}$ is said to be unitary related if every holomorphic function a_{i_j} which appears in any of the minimal relations (19) has no zero.

We also need the following Unit Theorem which is a consequence of the Borel's Lemma.

Unit Theorem. *Let $u_0,...,u_m$ be holomorphic functions without zeroes and $u_0 + \cdots + u_m = 1$. Suppose that no proper subsum $u_0 + \cdots + u_m - 1 = 0$ vanishes, then $u_0,...,u_m$ are all constants.*

The main result in this section is the following.

Theorem 4. *Let $L_i(z)(X) = \sum_{j=0}^{n} g_{ij}(z)X_j$, $1 \le i \le q$, $z \in \mathbb{C}$, where g_{ij} are holomorphic functions. Denote by $H_i(z)$ the corresponding moving hyperplane of $L_i(z)$, $1 \le i \le q$. Let $\mathcal{H} = \{H_1,...,H_q\}$ be unitary related. Then $\mathcal{H}$ is nondegenerate over Mero($\mathbb{C}$) if and only if there exist finitely many $(n+1)\times(n+1)$ invertible matrices with holomorphic functions as entries such that every holomorphic map omitting $\mathcal{H}(z)$ multiplied by one of the matrices is constant. In addition, this set of matrices depends only on the hyperplanes and can be determined effectively.*

Proof. Since the proof is completely parallel to the proof of Theorem 1, we will only reproduce the necessary parts. Let f_0,...,f_n be holomorphic functions on $\mathbb{C}$ without common zeroes and $(f_0,...,f_n)$ represents a holomorphic map f into $P^n(\mathbb{C})$ omitting $\mathcal{H}(z)$. Let $l_i = L_i(f_0,...,f_n)$. From (19) we have (after rearranging the index) the following unit equation

$$1+\frac{a_2l_2}{a_1l_1}+\cdots+\frac{a_ul_u}{a_1l_1}=0, \tag{20}$$

where no proper subsum vanishes. Then by the Unit Theorem, we have $\frac{a_il_i}{a_1l_1}$, $1\le i\le u$, is constant. The argument in the proof of Theorem 1 shows that there exist (after rearranging the index) $\{L_1,...,L_w\}\subset\mathcal{L}$ and holomorphic units $b_1,...,b_w,\ c_1,...,c_w$ such that $\dim(L_1,...,L_w)_K=n+1$, and $\frac{b_il_i}{c_il_1}$, $1\le i\le w$, is a nonzero constant. In addition, b_i and c_i , $1\le i\le w$, are coefficients of some minimal relations as (19). Therefore there are only finitely many such holomorphic units. After rearranging the index, we may assume that $L_1,...,L_{n+1}$ are linearly independent over Mero($\mathbb{C}$). Hence $(f_0(z),...,f_n(z))$ multiplied by

$$\begin{pmatrix} g_{10}(z), & \frac{b_2}{c_2}g_{20}(z), & \dots & \frac{b_{n+1}}{c_{n+1}}g_{n+1,0}(z)\\ \vdots & \ddots & \ddots & \vdots\\ g_{1n}(z), & \frac{b_2}{c_2}g_{2n}(z), & \dots & \frac{b_{n+1}}{c_{n+1}}g_{n+1,n}(z)\end{pmatrix} \tag{21}$$

is constant in $P^n(\mathrm{Hol}(\mathbb{C}))$. Since $L_1,...,L_{n+1}$ are linearly independent over Mero($\mathbb{C}$) and b_i, c_i are units, the matrix in (21) is invertible. It is also clear from the proof that the matrices used in (21) can be determined effectively and the number of the matrices is finite.

Conversely, if $\mathcal{H}$ is not nondegenerate over Mero($\mathbb{C}$), we assume that there exist finitely many $(n+1)\times(n+1)$ invertible matrices with holomorphic functions as entries such that every holomorphic map omitting $\mathcal{H}(z)$ multiplied by one of the matrices is constant. In the following proof, we refer to [Rub] for some basic results and definitions of Nevanlinna theory. Let $f(z)$ be a holomorphic function such that its characteristic function grows much rapidly than the characteristic function of the entries of the above matrices, i.e.

$$T(r,a)=o(T(r,f)),$$

for every entry $a(z)$ of the above matrices. Since $\mathcal{H}$ is not nondegenerate over Mero($\mathbb{C}$), there exists an $\mathcal{H}$-admissible subspace V of $P^n(\mathrm{Mero}(\mathbb{C}))$ of projective dimension greater than or equal to one such that $\mathcal{H}\cap V$ does not contain at least three distinct hyperplanes in $P^n(\mathrm{Mero}(\mathbb{C}))$ which are linearly dependent over Mero($\mathbb{C}$). We may assume, without loss of generality, that $V=P^n(\mathrm{Mero}(\mathbb{C}))$. Let $\mathcal{H}=\{H_1,...,H_q\}$. Then $q\le n+1$ and $H_1,...,H_q$ are linearly independent over Mero($\mathbb{C}$). We may assume that $H_1,...,H_q$ are the first q coordinate planes. Let $\exp^{[n]}f(z)$ is the function defined recursively by $\exp^{[1]}f(z)=e^{f(z)}$ and $\exp^{[j+1]}f(z)=e^{\exp^{[j]}f(z)}$. Then the holomorphic map represented by $(1,\exp^{[1]}f(z),\exp^{[2]}f(z),...,\exp^{[n]}f(z))$

omits $\mathcal{H}(z)$. If $A = (a_{ij})$ is one of the above matrices such that the product with this holomorphic map is constant, then

$$m_\beta \sum_{j=0}^{n} a_{\alpha j}(z)\exp^{[j]} f(z) = m_\alpha \sum_{j=0}^{n} a_{\beta j}(z)\exp^{[j]} f(z), \tag{22}$$

where $0 \le \alpha,\ \beta \le n$, m_α and m_β are constant. Since $T(r, a_{ij}) = o(T(r,f))$ and $T(r, \exp^{[j]} f) = o(T(r, \exp^{[j+1]} f))$, (22) implies that

$$m_\beta a_{\alpha j} = m_\alpha a_{\beta j}, \tag{23}$$

for $0 \le j \le n$. Therefore, the determinant of the matrix A is zero which contradicts the assumption that A is invertible. Therefore the proof is completed.

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