Discriminants, resultants, and their tropicalization

Course by Bernd Sturmfels - Notes by Silvia Adduci

2006

Contents

1 Introduction **3**

2 Newton polytopes and tropical varieties **3**

2.1 Polytopes 3

2.2 Newton polytope 6

2.3 Term orders and initial monomials 8

2.4 Tropical hypersurfaces and tropical varieties 9

2.5 Computing tropical varieties 12

2.5.1 Implementation in `GFan` 12

2.6 Valuations and Connectivity 12

2.7 Tropicalization of linear spaces 13

3 Discriminants & Resultants **14**

3.1 Introduction 14

3.2 The A-Discriminant 16

3.3 Computing the A-discriminant 17

3.4 Determinantal Varieties 18

3.5 Elliptic Curves 19

3.6 $2 \times 2 \times 2$-Hyperdeterminant 20

3.7 $2 \times 2 \times 2 \times 2$-Hyperdeterminant 21

3.8 Ge'lfand, Kapranov, Zelevinsky 21

4 Tropical Discriminants **22**

4.1 Elliptic curves revisited 23

4.2 Tropical Horn uniformization 24

4.3 Recovering the Newton polytope 25

4.4 Horn uniformization 29

5 Tropical Implicitization **30**

5.1 The problem of implicitization 30

5.2 Tropical implicitization 31

5.3 A simple test case: tropical implicitization of curves 32

5.4 How about for $d \geq 2$ unknowns? 33

5.5 Tropical implicitization of plane curves 34

6 References **35**

1 Introduction

The aim of this course is to introduce discriminants and resultants, in the sense of Gel'fand, Kapranov and Zelevinsky [6], with emphasis on the tropical approach which was developed by Dickenstein, Feichtner, and the lecturer [3]. This tropical approach in mathematics has gotten a lot of attention recently in combinatorics, algebraic geometry and related fields.

This notes are organized in four sections, corresponding to the four lectures of the course.

2 Newton polytopes and tropical varieties

2.1 Polytopes

A **polytope** $P \subset \mathbb{R}^n$ is the convex hull of finitely many points, or equivalently, the bounded intersection of finitely many closed halfspaces. The dimension of P is the smallest d such that P can be embedded in $\mathbb{R}^d$. A d-dimensional polytope is also called a d-polytope.

For example, a triangular prism is a 3-dimensional polytope:

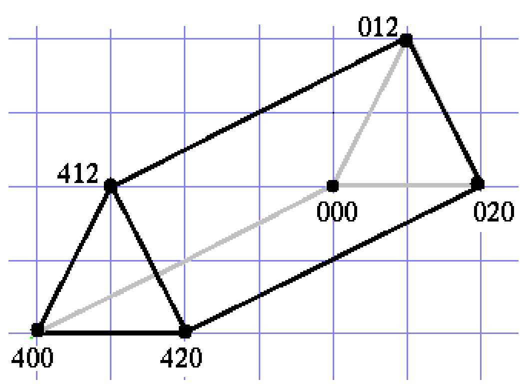


Figure 1: Triangular prism $\mathcal{T}$.

An affine hyperplane H is called a **supporting hyperplane** of P if

$$\begin{cases} H \cap P \neq \emptyset \\ P \text{ is fully contained in one of the two halfspaces defined by } H \end{cases}$$

Given a polytope P and a supporting hyperplane H, $F := P \cap H$ is called a **face** of P and is itself a polytope. A face is called a **facet** if it has the same dimension as the hyperplane.

For example, the facets of $\mathcal{T}$ are given by:

$$\begin{cases} x = 0 \\ x = 4 \\ z = 0 \\ 2y + z = 4 \\ 2y - z = 0 \end{cases}$$

Each vector $w \in \mathbb{R}^d$ defines a face of P as follows:

$$F_w(P) := \{u \in P \mid (u - v) \cdot w \leq 0 \ (\forall\ v \in P)\}.$$

For instance, let P be the unit square in $\mathbb{R}^2$ and let $w := (-1, 0)$. Then the face $F_{(-1,0)}$ is given by the points $(u_1, u_2) \ \in P$ satisfying

$$(u_1 - v_1, u_2 - v_2) \cdot (-1, 0) \ \leq \ 0$$

for every $(v_1, v_2) \ \in P$, *i.e.* :

$$v_1 \leq u_1 \ \ \forall\ 0 \leq v_1 \leq 1$$

Hence $u_1 = 1$ and $F_{(-1,0)} = \{(1, u_2) \mid 0 \leq u_2 \leq 1\}$.

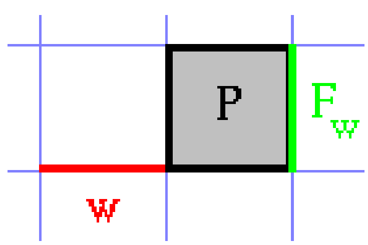


Figure 2: The unit square.

For F a given face of the polytope P, its **normal cone** is

$$\mathcal{N}_F(P) := \{w \in \mathbb{R}^d | \ F = F_w(P)\}.$$

In our previous example, let $F = \{x = 1\} \cap P$. The normal cone of F is given by the $w = (w_1, w_2) \ \in \mathbb{R}^2$ such that $F_w = \{x = 1\} \cap P$.
Claim:

$$\mathcal{N}_F(P) \ = \ \{(\lambda, 0) \mid \lambda < 0\}.$$

Let us check this.

$F_w = \{(u_1, \ u_2) \mid 0 \leq u_1, \ u_2 \leq 1 \text{ and } (v_1 - u_1)w_1 + (v_2 - u_2)w_2 \geq 0 \ \forall \ 0 \leq v_1, \ v_2 \leq 1\}$.

For F_w to agree with $\{x = 1\} \cap P$ it necessary that $u_1 = 1$. Hence $(1, u_2) \ \in F_w$ if and only if

$$\begin{cases} 0 \leq u_2 \leq 1 \\ (v_2 - u_2)w_2 \geq (1 - v_1)w_1 \ \forall \ 0 \leq v_1, \ v_2 \leq 1 \end{cases}$$

Note that $1 - v_1 \leq 0 \ \ \forall \ \ 0 \leq v_1 \leq 1$. So the only way the previous can hold is if $w_2 = 0$ and $w_1 \geq 0$.

The **normal fan** of P is the set of all normal cones:

$$\mathcal{N}(P) = \{\mathcal{N}_F(P) \mid F \text{is a face of } P\}.$$

The sum of two polytopes, called **Minkowski sum**, is another polytope:

$$P + Q := \{p + q \mid p \in P, \ q \in Q\}$$

with $\begin{cases} F_w(P + Q) = F_w(P) + F_w(Q) \\ \mathcal{N}(P + Q) \text{ is the common refinement of } \mathcal{N}(P) \text{ and } \mathcal{N}(Q). \end{cases}$

where the **common refinement** of the normal fans of two given two polytopes P and Q is defined to be

$$\mathcal{N}(P) \wedge \mathcal{N}(Q) := \{C \cap C' \mid C \in \mathcal{N}(P), \ C' \in \mathcal{N}(q)\}$$

For an example see figures (3) and (4).

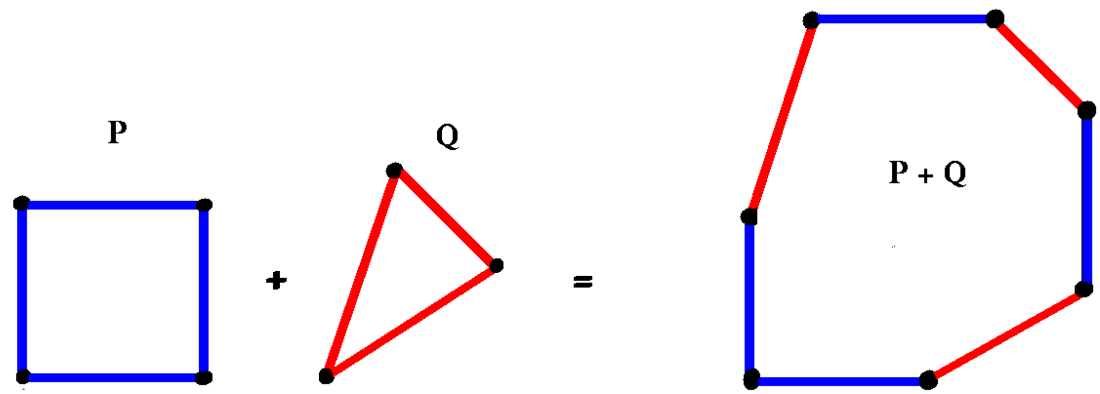


Figure 3: Minkowski sum.

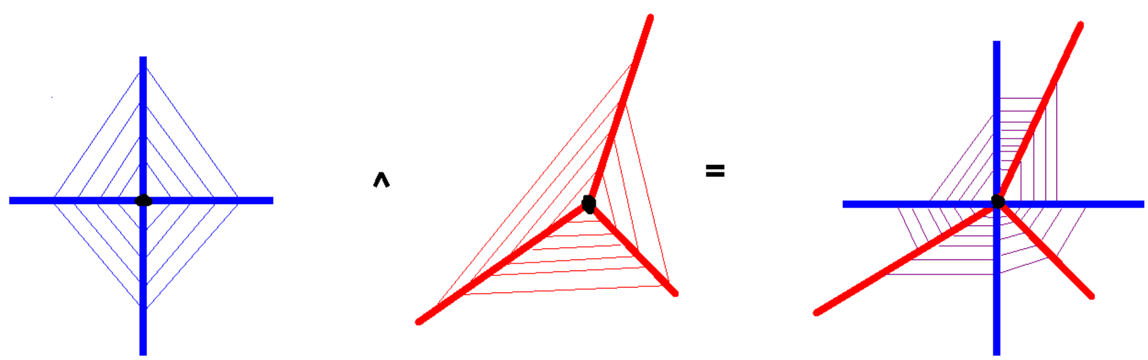


Figure 4: Normal fan of the sum $\mathcal{N}(P+Q)$.

2.2 Newton polytope

The Newton polytope is a tool for better understanding the behavior of polynomials. Let us consider the ring of formal Laurent series over the complex numbers and let $f \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$.

$$f = \sum_{\omega \in \mathbb{Z}^n} a_\omega x^\omega$$

where $\omega := (\omega_1, \ldots, \omega_n) \in \mathbb{Z}^n$ is an exponent vector and $x^\omega := x_1^{\omega_1} \ldots x_n^{\omega_n}$.
We define its **Newton polytope** $New(f)$ as the convex hull of all the exponent vectors ω such that $a_\omega \neq 0$:

$$New(f) := \mathrm{Conv}(\{\omega \in \mathbb{Z}^n \mid a_\omega \neq 0\})$$

The Newton polytope is also useful in questions of ramification for local fields, therefore in algebraic number theory. They have also been helpful in the study of ellliptic curves.

For example, let

$$f = 3x^2y - y^2 + 8x^2 + x. \tag{1}$$

In this case, the exponent vectors of the monomials of f are:

$$\{\omega \in \mathbb{Z}^2 \mid a_\omega \neq 0\} = \{(2,1); (0,2); (2,0); (1,0)\}.$$

Hence the Newton polytope of f is

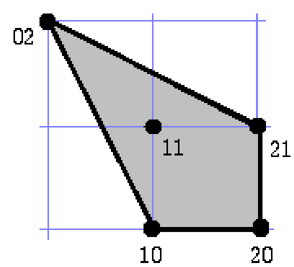


Figure 5: $New(f)$.

We can do algebra with polytopes: let f, g be polynomials. Then:

$$\begin{cases} New(f+g) = New(f) \cup New(g) \text{ (Unless there is cancellation of leading terms).} \\ New(f.g) = New(f) + New(g) \end{cases}$$

Example. Let $\begin{cases} f = 1 + t_1 + t_2 + t_1t_2 \\ g = 1 + t_1t_2^2 + t_1^2t_2 \end{cases}$

Then $\begin{cases} f+g &= 2 + t_1 + t_2 + t_1t_2 + t_1t_2^2 + t_1^2t_2 \\ fg &= 1 + t_1 + t_2 + t_1t_2 + t_1t_2^2 + t_1^2t_2 + 2t_1^2t_2^2 + t_1^3t_2 + t_1t_2^3 + t_1^2t_2^3 + t_1^3t_2^2 \end{cases}$

The Newton polytopes and the normal fans are as in figures (3) and (4).

2.3 Term orders and initial monomials

Given a polynomial f, the **initial form** $in(f)$ is the sum of the terms of maximal degree. For example: $in(x^2 + y + x + xy) = x^2 + xy$.
A term order for a Gröbner basis can be represented by a weight vector ([2]) $w = (w_1, ...w_d) \in \mathbb{R}^d$ in the following fashion. Let $u, v \in \mathbb{Z}^d$ be exponent vectors. Then

$$u >_w v \overset{def}{\Leftrightarrow} u \cdot w > v \cdot w.$$

We call $u \cdot w$ the w**-degree** or w**-weight** of x^u. Note that the monomial order defined by w is the same as the one defined by cw for $c > 0$, *i.e.*, this order does not depend on the length of w.

Given a polynomial f, the **initial form** with respect to a weight vector w (denoted $in_w(f)$) is the sum of the terms with maximal w-weight, *i.e*, those αx^u such that $u \cdot w$ is maximal. Note that this definition agrees with the previous one for $w = (1, ..., 1)$.
The initial form $in_w(f)$ satisfies

$$New(in_w(f)) = F_w(New(f)).$$

Let us consider again $f = x^2 + y + x + xy$ and let $w = (0, \sqrt{2})$.
Then $in_w(f) = x^2 + x$. So the Newton polytope of $in_w(f)$ is the horizontal segment in $\mathbb{R}^2$ joining $(2, 0)$ and $(1, 0)$. On the other hand, the Newton polytope of f is the quadrangle in figure (6).

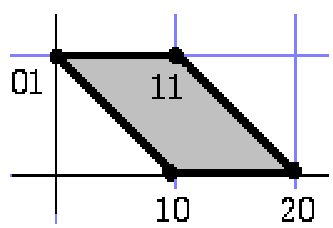


Figure 6: $New(in(f))$.

And the face of $New(f)$ induced by $w = (0, \sqrt{2})$ is:

$$\begin{aligned} F_w &= \{u \in New(f) \mid (u - v) \cdot (0, \sqrt{2}) \leq 0 \ (\forall\ v \in New)\}. \\ &= \{u \in New(f) \mid u_2 \leq v_2 \ (\forall\ v \in New)\}. \\ &= \{u \in New(f) \mid u_2 = 0\} \\ &= New(f) \cap \{y = 0\} \end{aligned}$$

so both of them agree.

2.4 Tropical hypersurfaces and tropical varieties

Let $f \in \mathbb{C}[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ and let w be a weight vector. If w is generic then $in_w(f)$ is a monomial. We define the **tropical hyperfurface** or the **tropicalization** of f as

$$\begin{aligned} \mathcal{T}(f) &:= \{w \in \mathbb{R}^d | \ in_w(f) \textit{ is not a monomial } \} \\ &= \{w \in \mathbb{R}^d | \ F_w(New(f)) \textit{ has dimension } \geq 1\} \end{aligned}$$

Let us recall our example (1):

$$f = 3x^2y - y^2 + 8x^2 + x.$$

In this case, the tropical hypersurface of f given by:

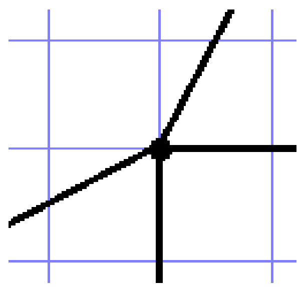


Figure 7: $\mathcal{T}(f)$: 4 half rays in $\mathbb{R}^2$.

But 4 half rays are equivalent to their 4 intersections with the unit circle. Hence, our tropical hypersurface is as in figure (8).

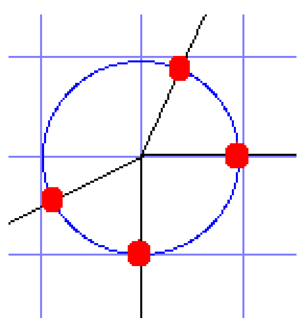


Figure 8: $\mathcal{T}(f)$: 4 points in $\mathbb{R}^2$.

A **tropical prevariety** is a finite intersection of tropical hypersurfaces in $\mathbb{R}^n$.

Let $I \subset \mathbb{C}[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ be an ideal. Its **tropical variety** $\mathcal{T}(I)$ is the intersection of the tropical hypersurfaces $\mathcal{T}(f)$ where f runs over all polynomials $f \in I$, ie:

$$\mathcal{T}(I) = \bigcap_{f \in I} \mathcal{T}(f).$$

We may have prevarieties that are not varieties as in the following example. However, the converse is indeed true, as the tropical Hilbert basis theorem states.

Example. Let $d = 3$ and $\begin{cases} f_1 = t_1 + t_2 + t_3 + 1 \\ f_2 = t_1 + t_2 + 2t_3 \end{cases}$

The Newton polytope and the tropical prevarieties of f_1 and f_2 are given by figures (9) and (10) respectively.

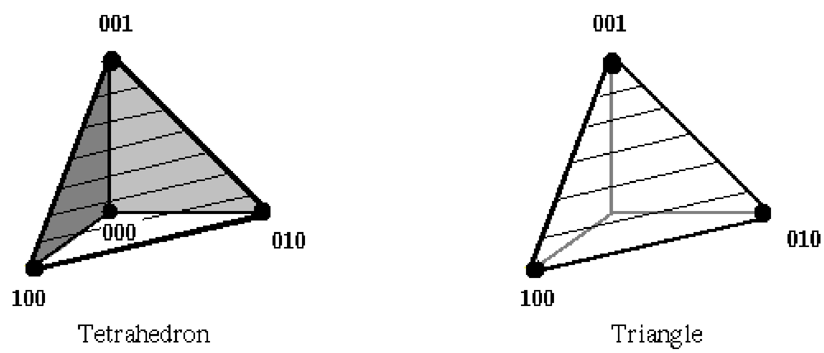


Figure 9: $New(f_1)$ and $New(f_2)$.

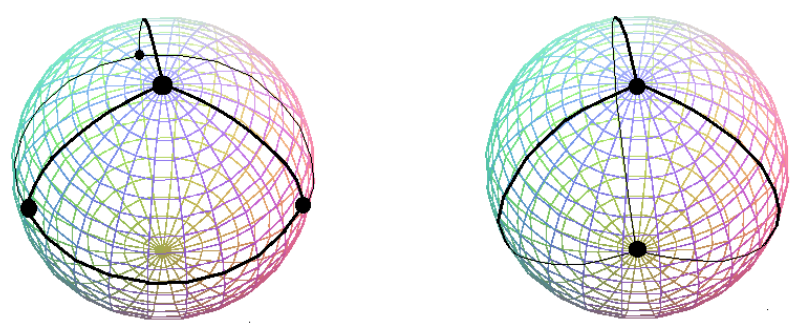


Figure 10: $\mathcal{T}(f_1)$ and $cT(f_2)$.

But the intersection $\mathcal{T}(f_1) \cap \mathcal{T}(f_2)$ is not a variety, as we can clearly see in figure (11).

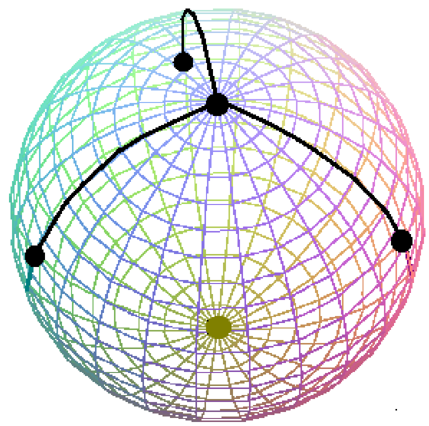


Figure 11: $\mathcal{T}(f_1) \cap \mathcal{T}(f_2)$.

Theorem 2.1 *(Tropical Hilbert Basis)*
Every tropical variety is a tropical prevariety.

Proof.
Theorem 2.9 in [1].

Note that, for any two ideals I and J we have $\mathcal{T}(I+J) \subseteq \mathcal{T}(I) \cap \mathcal{T}(J)$. A sufficient condition for the equality to hold is given by the following theorem.

Lemma 2.2 (Transverse Intersection Lemma)
Let I and J be ideals in $\mathbb{C}[x_1, \ldots, x_n]$ whose tropical varieties $\mathcal{T}(I)$ and $\mathcal{T}(J)$ meet transversally at a point $w \in \mathbb{R}^n$. Then $w \in \mathcal{T}(I+J)$.

By "**meet transversely**" we mean that if F and G are the cones of $\mathcal{T}(I)$ and $\mathcal{T}(J)$ which contain w in their relative interior, then $\mathbb{R}F + \mathbb{R}G = \mathbb{R}^n$.

This lemma implies that any transverse intersection of tropical varieties is a tropical variety. In particular, any transverse intersection of tropical hypersurfaces is a tropical variety, and such a tropical variety is defined by an ideal which is a complete intersection in the commutative algebra sense.

Proof. We have $\mathcal{T}(I) \cap \mathcal{T}(J) = \bigcap_{f \in I} \mathcal{T}(f) \cap \bigcap_{f \in J} \mathcal{T}(f) = \bigcap_{f \in I \cup J} \mathcal{T}(f)$. Clearly, this contains $\mathcal{T}(I+J) = \bigcap_{f \in I+J} \mathcal{T}(f)$. If $\mathcal{T}(I)$ and $\mathcal{T}(J)$ intersect transversally and w is a point of $\mathcal{T}(I) \cap \mathcal{T}(J)$ other than the origin then the preceding lemma tells us that $w \in \mathcal{T}(I+J)$. Thus $\mathcal{T}(I+J)$ contains every point of $\mathcal{T}(I) \cap \mathcal{T}(J)$ except possibly the origin. In particular, $\mathcal{T}(I+J)$ is not empty. Every nonempty fan contains the origin, so we see that the origin is in $\mathcal{T}(I+J)$ as well.

□

2.5 Computing tropical varieties

(math.AG/0507563 - T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, R. Thomas. [1].)

This paper presents algorithmic tools for computing the tropical variety of a d-dimensional prime ideal in a polynomial ring with complex coefficients.

It discusses as well the implementation of these tools in the Gröbner fan software `Gfan`.

2.5.1 Implementation in `GFan`

`Input:` A homogeneous ideal $I \in \mathbb{C}[t_0, t_1 ..., t_d]$.
`Output:` The fan $\mathcal{T}(I)$, represented as a spherical polyhedral complex.

Example:

`Input:` $I =< 1 + t_1 + t_2 + t_3,\ t_1 + t_2 + 2t_3 >$
`Output:` Three end points as in the graph in figure (11).

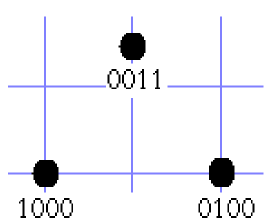


Figure 12: Three points.

2.6 Valuations and Connectivity

Let $K = \mathbb{C}\{\{\ \varepsilon\}\}$ be the Puiseux series ring.

The valuation

$$\nu : K^* \to \mathbb{Q}$$

induces a map

$$\nu : (K^*)^d \to \mathbb{Q}^d \hookrightarrow \mathbb{R}^d$$

Theorem 2.3 *Let $I \in \mathbb{C}[t_1^{\pm 1}, ..., t_d^{\pm 1}]$ be an ideal. Then the its tropical variety $\mathcal{T}(I)$ equals the closure of the image of the classical variety $\mathcal{V}(I) \in (K^*)^d$ under the valuation ν.*
Furthermore, if I is prime of dimension k then $\mathcal{T}(I)$ is pure of dimension k and connected in codimension 1.

This was first proved by Kapranov ([4]) in the case when I is a principal ideal. The general case was first stated in [9] and proved in [10].

2.7 Tropicalization of linear spaces

Before we get started let us clarify that the adjective *tropical* is given in honor of the Brazilian mathematician Imre Simon, who pioneered the field. It simply reflects the French view on Brazil (as it was coined by a Frenchman). Besides that, it has no deeper meaning.
Let I be generated by τ linearly independent linear forms in $\mathbb{C}[t_0, t_1 ..., t_d]$. Then we can easily compute $\mathcal{T}(I)$ from the matroid of I, which has rank τ on $\{0, 1, ...d\}$.

Example. If the linear forms are generic, then $\mathcal{T}(I)$ is the $(d - \tau + 1)$-dimensional fan represented by the $(d - \tau - 1)$-skeleton of the simplex on $\{0, 1, ...d\}$.

Running example:

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 \\ 53 & 47 & 43 & 31 & 37 & 31 & 29 & 23 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The rows represent $\tau = 4$ linear forms in $d + 1 = 8$ unknowns $t_0, t_1, ...t_7$, generating a linear ideal I. To solve this system, we no longer need to use Gauss elimination. We now can use `Gfan`.
The tropical linear space $\mathcal{T}(I)$ is a 2-dimensional simplicial complex with 10 vertices and 60 triangles.

3 Discriminants & Resultants

3.1 Introduction

For a quadratic polynomial $f(t) = x_1 + x_2 t + x_3 t^2$ we might remember from high-school the formula for the roots:

$$\frac{-x_2 \pm \sqrt{x_2^2 - 4x_3x_1}}{2x_3}.$$

Therefore f has a double root if and only if its coefficients satisfy

$$x_2^2 - 4x_3x_1 = 0.$$

The polynomial in three variables defining this equation is called **discriminant** and is denoted by Δ :

$$\Delta(x_1, x_2, x_3) = x_2^2 - 4x_3x_1.$$

For a cubic univariate polynomial

$$f(t) = x_1 + x_2 t + x_3 t^2 + x_4 t^3$$

the discriminant equals

$$\Delta = 27x_1^2x_4^2 - 18x_1x_2x_3x_4 + 4x_1x_3^3 + 4x_2^3x_4 - x_2^2x_3^2$$

The Newton polytope of Δ is the quadrangle in figure (13).

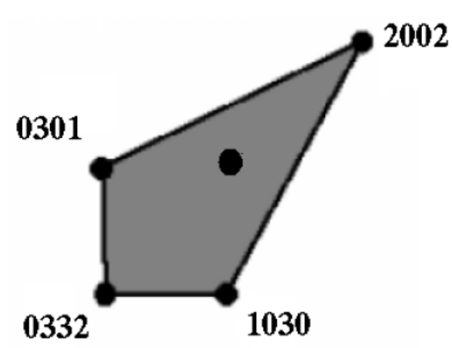


Figure 13: New(Δ).

We want a more general theory. Let f be a d-variate polynomial

$$f(t) = \sum_{j=1}^{n} x_j t_1^{a_{1j}} t_2^{a_{2j}} ... t_d^{a_{dj}}.$$

We want to know the coefficients $x_1, x_2, ..., x_n$ that make f to have a multiple root. By **multiple root** we mean a point $t = (t_1, ..., t_d) \in (\mathbb{C}^*)^d$ such that

$$\begin{cases} f(t) &= 0 \\ \dfrac{\partial f}{\partial t_j}(t) &= 0 \ \forall\ 1 \leq j \leq d \end{cases}$$

But the coefficients $x_1, x_2, ..., x_n$ will have some obvious restrictions. For instance, they cannot be all zero, since in that case the polynomial would be $f = 0$, and this case has no interest for us. So we will restrict our search to points $(x_1, x_2, ..., x_n)$ with at least one coordinate being not zero. That is equivalent to taking points $[x_1 : x_2 : ... : x_n]$ in the projective space $\mathbb{P}_{\mathbb{C}}^{n-1}$.

Note that a polynomial having a multiple root is equivalent as the hypersurface $\{f = 0\}$ having a singularity.

It will be useful to use the following representation. We refer to $f(t) = \sum_{j=1}^{n} x_j t_1^{a_{1j}} t_2^{a_{2j}} ... t_d^{a_{dj}}$ as a *family* of polynomials (not just *one* polynomial) because we are considering the coefficients x_j as variables. So this family is actually defined by the monomials that appear in f. But the monomials are uniquely determined by the exponent vectors, which we will arrange in a matrix by putting them as columns.

For example, the family of polynomials

$$f(t) = x_1 t_2^2 + x_2 t_1 t_2 + x_3 t_1^2$$

has vector exponents

$$\{(0, 2)(1, 1)(2, 0)\}$$

therefore is equivalent to the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

Also, each polynomial defines a hypersurface (given by its zeros), so a family of polynomials will define a family of hypersurfaces. Therefore the matrix will also represents this family.

3.2 The A-Discriminant

Let $A \in \mathbb{Z}^{d \times n}$ be a matrix satisfying the conditions:

$$\begin{cases} rank(A) = d \text{ (linearly independent rows).} \\ (1, ..., 1) \ \in \ \text{ the row space of } A \end{cases}$$

Such a matrix A represents a family of hypersurfaces in $(\mathbb{C}^*)^d$ defined by the Laurent polynomial

$$f(t) = \sum_{j=1}^{n} x_j t_1^{a_{1j}} t_2^{a_{2j}} ... t_d^{a_{dj}}.$$

Consider the set of points $[x_1 : x_2 : ... : x_n] \in \mathbb{P}_{\mathbb{C}}^{n-1}$ such that the hypersurface $\{f = 0\}$ has a singular point in $(\mathbb{C}^*)^d$.

The closure of this set is an irreducible variety in $\mathbb{P}_{\mathbb{C}}^{n-1}$ denoted Δ_A and called the A-**discriminant**:

$$\Delta_A := \overline{\{[x_1 : x_2 : ... : x_n] \in \mathbb{P}_{\mathbb{C}}^{n-1} \mid \{f = 0\} \ \text{ has a singular point in } (\mathbb{C}^*)^d\}}$$

Recall that an **irreducible variety** is a variety given by a prime ideal [6].

Δ_A is often a hypersurface defined by an irreducible polynomial over $\mathbb{Z}$ ([6]). We are only interested in this case.

We will usually refer to the A-discriminant indistinctly either as the closure of the previous set or as the irreducible polynomial of content 1 that defines it. Note that we cannot distinguish between this polynomial or its negative.

Example. Discriminant of a binary form: $d = 2, n = 3$.

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

As we have seen before, this matrix represents the family of bivariate polynomials:

$$f = x_1 t_2^2 + x_2 t_1 t_2 + x_3 t_1^2$$

which has A-discriminant

$$\Delta_A = x_2^2 - 4x_1 x_3.$$

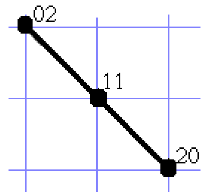


Figure 14: The columns of A determine $New(f)$.

3.3 Computing the A-discriminant

We can compute the A-discriminant by means of the partial derivatives using `Macaulay 2`. Consider the ideal generated by the partial derivatives of f.

$$I = < \frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2}, ..., \frac{\partial f}{\partial t_d} >$$

I is a polynomial ideal in $d+n$ variables: $t_1, ...t_d \succ x_1, ...x_n$.
Eliminate $t_1, ...t_d$ to get $\Delta_A(x_1, ...x_n)$.
Example. The discriminant of the rectangle and the Sylvester resultant.
Let us compute the A-discriminant of the polytope in figure (15).

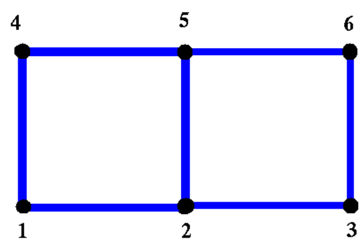


Figure 15: Rectangle.

There are 6 points in dimension3 (we always use homogeneous coordinates), hence $d = 3$ and $n = 6$.
Arranging the vector exponents in a matrix we obtain:

$$A = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and the family of polynomials represented by A is:

$$f = x_1t_2 + x_2t_1t_2 + x_3t_1^2t_2 + x_4t_3 + x_5t_1t_3 + x_6t_1^2t_3.$$

To compute Δ_A we take derivatives:

$$\begin{cases} \dfrac{\partial f}{\partial t_1} & = & x_2t_2 + 2x_3t_1t_2 + x_5t_3 + 2x_6t_1t_3 \\ \\ \dfrac{\partial f}{\partial t_2} & = & x_1 + x_2t_1 + x_3t_1^2 \\ \\ \dfrac{\partial f}{\partial t_3} & = & x_4 + x_5t_1 + x_6t_1^2 \end{cases}$$

Note that $\dfrac{\partial f}{\partial t_2}$ and $\dfrac{\partial f}{\partial t_3}$ are univariate polynomials, both depending on the same variable t_1. Therefore, it makes sense to compute their resultant (with respect to t_1 of course). This will eliminate the variable t_1, obtaining a polynomial on the 6 variables $x_1, ..., x_6$ which vanishes iff $x = (x_1, ..., x_6)$ is a common root of $\dfrac{\partial f}{\partial t_2}$ and $\dfrac{\partial f}{\partial t_3}$.

So let us compute the resultant.

$$\begin{array}{rcl} \Delta_A & = & Res_{t_1}(\dfrac{\partial f}{\partial t_2}, \dfrac{\partial f}{\partial t_3}) \\ & = & Res_{t_1}(x_1 + x_2t_1 + x_3t_1^2,\ x_4 + x_5t_1 + x_6t_1^2) \\ & = & det \begin{pmatrix} x_1 & x_2 & x_3 & 0 \\ o & x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 & 0 \\ 0 & x_4 & x_5 & x_6 \end{pmatrix} \end{array}$$

This is a polynomial of degree 4 in 6 unknowns having 7 terms.

3.4 Determinantal Varieties

Let $d = 4, n = 6$ and:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

This configuration represents the product of two simplices that forms the triangular prism. It also gives a family of bilinear forms:

$$f = (t_1,\ t_2) \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix} \begin{pmatrix} t_3 \\ t_4 \\ t_5 \end{pmatrix}$$

The A-discriminant Δ_A is the codimension two variety of all rank one matrices

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{pmatrix}.$$

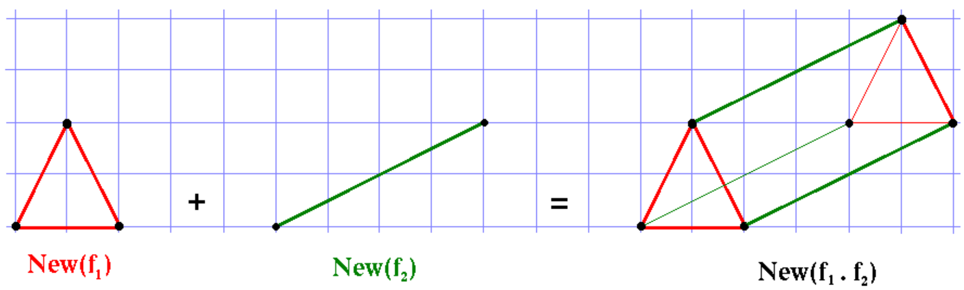


Figure 16: Triangular prism.

3.5 Elliptic Curves

Let $d = 3, n = 10$ and

$$A = \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}$$

$$f = \text{homogeneous cubic polynomial in } t_1,\ t_2,\ t_3.$$

The discriminant Δ_A is a polynomial of degree 12 in 10 unknowns and 2040 monomials which vanishes iff the plane cubic $\{f = 0\}$ has a singular point.

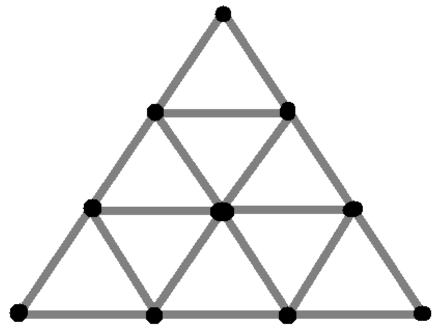


Figure 17: $n = 10$ vertices.

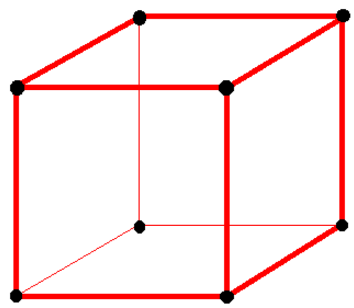


Figure 18: 3-cube.

3.6 $2 \times 2 \times 2$-Hyperdeterminant

$$f = \sum_{i,j,k=0}^{1} x_{ijk} t_i^{(1)} t_j^{(2)} t_k^{(3)}$$

$$A = \text{the 3-cube}$$

$$\begin{aligned}
\Delta_A = \; & 4x_{000}x_{011}x_{101}x_{110} + 4x_{001}x_{100}x_{010}x_{111} + \\
& x_{000}^2 x_{111}^2 + x_{001}^2 x_{110}^2 + x_{010}^2 x_{101}^2 + x_{100}^2 x_{011}^2 + \\
& -2x_{000}x_{111}x_{001}x_{110} - 2x_{000}x_{111}x_{010}x_{101} + \\
& -2x_{000}x_{111}x_{100}x_{011} - 2x_{001}x_{110}x_{010}x_{101} + \\
& -2x_{001}x_{110}x_{100}x_{011} - 2x_{010}x_{101}x_{100}x_{011}
\end{aligned}$$

Physicists call this the **tangle**.

3.7 $2 \times 2 \times 2 \times 2$-Hyperdeterminant

Here we consider multilinear forms in four groups of variables.

$$
\begin{aligned}
f &= \sum_{i,j,k,l=0}^{1} x_{ijkl} t_i^{(1)} t_j^{(2)} t_k^{(3)} t_l^{(4)} \\
A &= \text{the 4-cube} \\
\Delta_A &= \text{hyperdeterminant of the tensor } (x_{ijkl}).
\end{aligned}
$$

The A-discriminant has degree 24 and is the sum of 2,894,276 monomials, out of which only 25,448 are of maximal degree. Hence the Newton polytope of Δ_A has 25,448 vertices. It has dimension 11.

In order to count the vertices, we count the zero dimensional faces of the Newton polytope . We do this by randomly picking weight vectors w and minimizing the linear function they define. ([5]).

Note that in all these examples the underlying toric variety χ_A is smooth. This has all been solved by Gel'fand-Kapranov-Zelevinsky. ([6]).

3.8 Ge'lfand, Kapranov, Zelevinsky

We refer to [6] as the basic book for this work. And by basic we do not mean easy, on the contrary, the book is quite advanced. It is an essential and indispensable tool for the study of the material in this course.

Key results in this book:

• All classical resultants and discriminants are A-discriminants. (Mild exaggeration).

• The Newton polytope of Δ_A is a Minkowski summand of the secondary polytope of Δ_A.([6]$chapter 7$).

• An alternating degree formula for Δ_A in the special case when the toric variety χ_A is smooth.

• Techniques are quite advanced and give little information when χ_A is not smooth or $codim(\chi_A) > 1$.

4 Tropical Discriminants

(math.AG/0510126 - A. Dickenstein, E.M. Feichtner, and B. Sturmfels. [3].)

The tropical A-discriminant is shown to coincide with the Minkowski sum of the row space of A and the tropicalization of the kernel of A. This leads to an explicit positive formula for the extreme monomials of any A-discriminant, and to a combinatorial rule for deciding when two regular triangulations of A correspond to the same monomial of the A-discriminant.

This paper offers a strategy to obtain some relevant information about the A-discriminant. Usually the A-discriminant has many terms; and is impossible to compute. But even if it were possible, it would not be of much interest. However, there is some information of the A-discriminant that we can effectively obtain and that is indeed of much interest.

We can find in [3] an explicit combinatorial description of the tropicalization of the A-discriminant Δ_A for any integer matrix A. If $codim(\Delta_A) = 1$ this gives an efficient method for computing the Newton polytope of Δ_A.

The main contribution of [3] is that A can be *any* matrix, not only those that come from smooth varieties. In fact, in the smooth case [6] is enough, and [3] does not add too much.

It is worth mentioning that [3] is self contained, and there is no real need to read [6] before.

Let us see a simple example. Let us think of [3] as a black box, and feed this black box with an input matrix A representing a binary cubic form, namely $x_1t_1^3 + x_2t_1^2t_2 + x_3t_1t_2^2 + x_4t_2^3$.

`Input`:

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

The methods in [3] allow us to compute the Newton polytope of the A-discriminant and the coefficients of the leading monomials as well.

`Output`: $New(\Delta_A)$ (figure 19).

The information we are interested in, and that can be obtained by [3], is compressed in the *f-vector*. Given a d-dimensional polytope P, its ***f*-vector** is a d-dimensional vector whose j^{th} coordinate is the number of $(j+1)$-dimensional faces of P, for $0 \leq j \leq\ d-1$

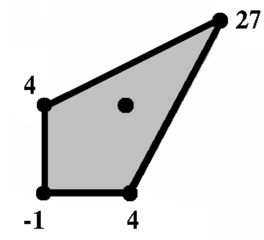


Figure 19: $New(\Delta_A)$.

4.1 Elliptic curves revisited

`Input:` Let us feed our "black box" with a family of elliptic curves represented by a matrix A on 10 columns labeled a, b, ... , j.
A : figure (20).

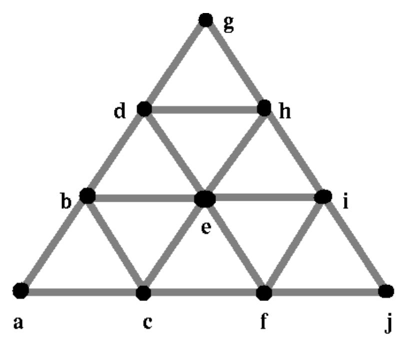


Figure 20: A.

Specifically, our input matrix is

$$A = \begin{pmatrix} 3 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 \end{pmatrix}$$

`Output:` the Newton polytope of Δ_A, which is given by the points $(t_1, t_2, ..., t_{10}) \in \mathbb{R}^{10}$ such that

$$A \begin{pmatrix} t_1 \\ \vdots \\ t_{10} \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$$

and satisfying the system $\mathcal{S}$:

$$\mathcal{S} = \begin{cases} t_1,\ t_2,\ \dots,t_{10} & \geq & 0 \\ 2t_1 + t_2 + t_3 & \geq & 2 \\ 2t_{10} + t_6 + t_9 & \geq & 2 \\ 2t_7 + t_4 + t_8 & \geq & 2 \\ t_2 + t_4 + t_5 & \leq & 9 \\ t_5 + t_8 + t_9 & \leq & 9 \\ t_3 + t_5 + t_6 & \leq & 9 \\ 2t_1 + t_2 + t_3 + t_4 + t_7 & \geq & 3 \\ 2t_7 + t_4 + t_8 + t_9 + t_{10} & \geq & 3 \end{cases}$$

Note that $(4,4,4)$ is a fancy way to say that the degree is 12. We know this because A is smooth. [6]

Therefore $New(\Delta_A)$ is given by 18 linear inequalities, *i.e.*, 18 half spaces. Since they are all independent, $New(\Delta_A)$ has 18 facets, as we can see in its 7-dimensional f-vector

$$(133,\ 513,\ 846,\ 764,\ 402,\ 126,\ 18).$$

We read from the f-vector that $New(\Delta_A)$ has 133 vertices. This means that out of all all the terms (2040, corresponding to the lattice points) only 133 are of maximal w-degree for some weight vector w. Δ_A also has 513 edges, 842 2-dimensional faces and so on, and 18 6-dimensional faces which are the facets, since $New(\Delta_A)$ is a 7-dimensional. The total number of faces is therefore 133 + 513 + ... + 18 + 1 = 2803. (The last 1 we add corresponds to the 7-dimensional face, namely the whole polytope itself).

The 18 facets come in 4 classes corresponding to he following coarsest subdivisions of A. (Figure 21).

4.2 Tropical Horn uniformization

Recall that the kernel of A is a linear variety in $\mathbb{P}^{n-1}_{\mathbb{C}}$ and its tropicalization $\mathcal{T}(ker\ A)$ can be computed from the matroid of A. The main theorem in which [3] relies is the following.

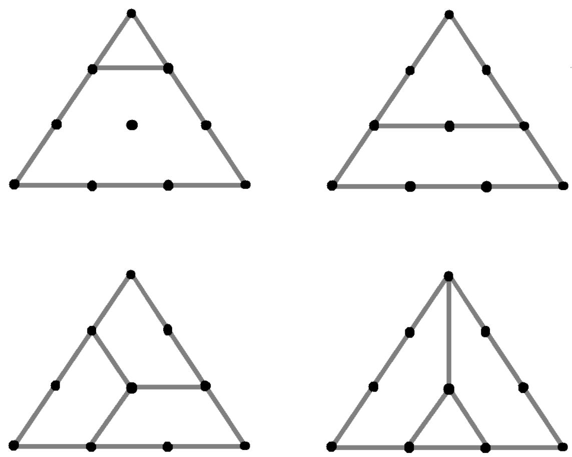


Figure 21: Coarsest subdivisions of A.

Theorem 4.1 *The tropical A-discriminant is the sum of the linear space spanned by the rows of A and the tropical linear space determined by ker A :*

$$\mathcal{T}(\Delta_A) = \textit{ row space } (A) + \mathcal{T}(\textit{ker } A)$$

Note that the dimensions make sense, since the row space of A has dimension $d-1$ and the kernel of A has dimension $n-d$ and $n-d-1$ after tropicalizing.

This implies that the space spanned by the rows of A and the tropicalization of the kernel of A are in general position if and only if the dimension of $\mathcal{T}(\Delta_A)$ is $n-2$. Otherwise, they do not intersect transversally.

On the other hand, we know that since the codimension of Δ_A is 1, the dimension of $\mathcal{T}(\Delta_A)$ has to be $n-2$.

4.3 Recovering the Newton polytope

Suppose Δ_A is a hypersurface. (We already have a formula that gives a test for this).

Theorem 4.2 *Fix $\omega \in \mathbb{R}^n$ generic. The exponent of x_i in $in_\omega(\Delta_A)$ equals the number of intersection points of the tropical discriminant with the halfray $\omega + \mathbb{R}_{\geq 0} \cdot e_i$ counting multiplicities.*

Let us just focus on monomials, not coefficients.

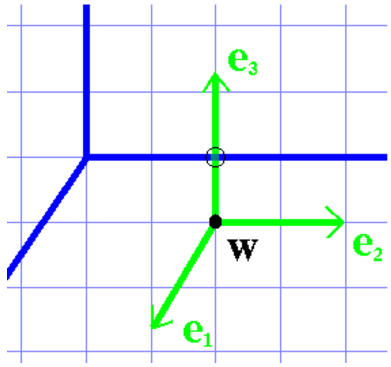


Figure 22: Counting intersections.

Note that we are taking half rays because in tropical geometry, a line is a union of half rays. In classical algebraic geometry, what we would be doing here is drawing lines and counting intersections. In tropical geometry we do the same, but with tropical lines.

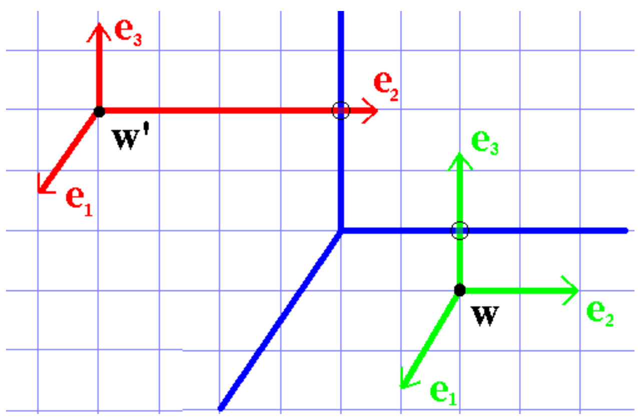


Figure 23: The number of intersections is independent of the choice of w.

Example. Let $d = 3, n = 6$ and

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

$\mathcal{T}(ker\ A)$ is a 3-dimensional fan in 6-dimensional space, and $\mathcal{T}(\Delta_A)$ is the normal fan of the Newton polytope of Δ_A.

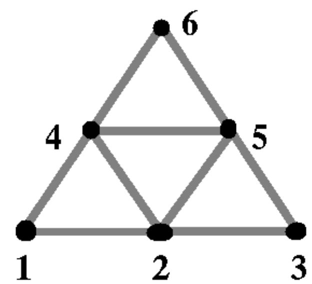


Figure 24: A.

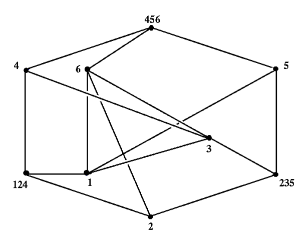


Figure 25: $\mathcal{T}(ker\ A)$.

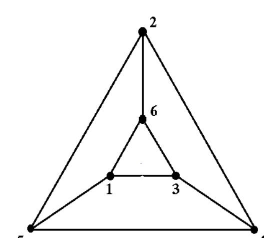


Figure 26: $\mathcal{T}(\Delta_A)$.

Our running example.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 \\ 53 & 47 & 43 & 41 & 37 & 31 & 29 & 23 \end{pmatrix}$$

is row equivalent to

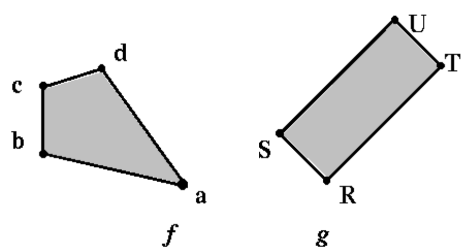
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 3 & 5 & 0 & 2 & 6 & 8 \end{pmatrix}$$

This Cayley matrix represents a system of two equations in two unknowns.

$$\begin{aligned} f(x,y) &= ax^2 + by + cy^3 + dxy^5 \\ g(x,y) &= Rx + Sy^2 + Tx^3y^6 + Ux^2y^8 \end{aligned}$$

For generic coefficients a, b, c, d, R, S, T, U; the system $f = g = 0$ has 24 solutions $(x, y) \in (\mathbb{C}^*)^2$.

Mixed area of



The discriminant Δ_A is the irreducible polynomial in a, b, c, d, R, S, T, U which vanishes whenever the system

$$f = g = 0$$

has a solution $(x, y) \in (\mathbb{C}^*)^2$ of multiplicity 2 or more.

4.4 Horn uniformization

The **Horn uniformization** is a parametric representation of the A-discriminant:

$$\begin{cases} a &= -2c_4\ t_1\ t_3^2 \\ b &= (c_2 - 2c_3 + c_4)\ t_1\ t_4 \\ c &= (c_2 + 3c_3)\ t_1\ t_4^3 \\ d &= (-2c_2 - c_3 + c_4)\ t_1\ t_3\ t_4^5 \\ R &= (c_1 + c_4)\ t_2\ t_3 \\ S &= (-c_1 - c_2 - c_4)\ t_2\ t_4^2 \\ T &= (-c_1 + c_3 + 2c_4)\ t_2\ t_3^3\ t_4^6 \\ U &= (c_1 + c_2 - c_3 - 2c_4)\ t_2\ t_3^2\ t_4^8 \end{cases}$$

We can implicitize this map $\mathbb{C}^8 \to \mathbb{C}^8$ by doing it tropically first.

The Newton polytope of Δ_A has dimension 4 and f-vector $(74,\ 158,\ 110,\ 26)$. The 74 extreme monomials of Δ_A are

$$\begin{gathered} a^{10}\ b^{18}\ c^{18}\ d^1\ R^0\ S^{18}\ T^{29}\ U^2 \\ a^{10}\ b^{18}\ c^8\ d^{11}\ R^0\ S^{22}\ T^{27}\ U^0 \\ \dots \\ a^{42}\ b^0\ c^2\ d^3 R^{11}\ S^2\ T^{26}\ U^{10} \end{gathered}$$

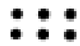
The total number of lattice points in this polytope is

$$\begin{array}{rr} & 74 \\ + & 81 \\ + & 753 \\ + & 4082 \\ + & 16,186 \\ & --- \\ & 21,176 \end{array}$$

The method used by [3] basically follows the next steps:

1. Start with the 60 triangles representing the 3-dimensional tropical linear space $\mathcal{T}(ker\ A)$.

2. Take its image under the linear map $\mathbb{R}^8 \to \mathit{coker}\ (A)$ to 48 immersed cones.



3. The result is a 3-dimensional fan with 158 cones on 26 rays.

4. This is the tropical hypersurface $\mathcal{T}(\Delta_A)$.

5. Now reconstruct the Newton polytope .

5 Tropical Implicitization

(Joint work with Jenia Tevelev & Josephine Yu).

`Input`: parameterization of the Horn uniformization of the discriminant of a polynomial in one variable of degree three.

$$\begin{cases} x_1 &= c_1\ t_1^3 \\ x_2 &= (-2c_1 + c_2)\ t_1^2\ t_2 \\ x_3 &= (c_1 - 2c_2)\ t_1\ t_2^2 \\ x_4 &= c_2\ t_2^3 \end{cases}$$

`Output`: the Newton polytope of the implicit equation.

5.1 The problem of implicitization

Given n polynomials $f_1, ...f_n$ in d unknowns $t = (t_1, ..., t_d)$, let us consider the ring map

$$\begin{array}{ccc} \mathbb{C}[x_1, ..., x_n] & \xrightarrow{\varphi} & \mathbb{C}[t_1, ..., t_d] \\ x_i & \mapsto & f_i(t) \end{array}$$

The kernel of this map is a *prime* ideal $I\ \in \mathbb{C}[x_1, ..., x_n]$.

The problem of implicitization is to find a generating set or a Gröbner basis for the kernel.

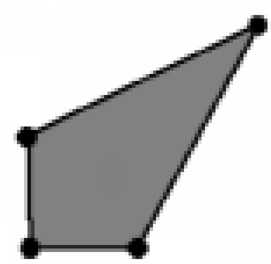


Figure 27: Newton polytope of the implicit equation.

However, the softwares we know so far do not seem to work for this. Implicitization is a tough problem. Is the typical case when after two months of running the program you abort the process. Next we will see an alternative.

5.2 Tropical implicitization

Even if we are not able to get $\mathcal{T}(I)$, we can try to compute the tropical variety $\mathcal{T}(I)$ directly from $f_1, ...f_n$.

We have seen that for the case of A discriminants this succeeds. Now we would like to know what is the level of generality, for what other polynomial maps will this work.

Suppose we succeed with the tropicalization of the kernel. The next obvious question would be if we can recover I from $\mathcal{T}(I)$.

The answer is predictable: "not quite." But we can still recover some valuable information. As in previous sections we learnt how to recover the Newton polytope and the valuable information it carries, now we will recover the **Chow polytope** of I. For a complete definition of Chow polytope see [6]. Roughly speaking, is a generalization of the idea of the Newton polytope from hypersurfaces to algebraic subvarieties of arbitrary dimension.

Example.

$$\begin{cases} x_1 &= t_1\ t_2\ (t_1^4 - t_2^4) \\ x_2 &= \text{ Hessian } (x_1(t)) \\ x_3 &= \text{ Jacobian } (x_1(t),\ x_2(t)) \end{cases}$$

The implicit equation for this map $\mathbb{C}^2 \longrightarrow \mathbb{C}^3$ is a function $g(x_1,\ x_2,\ x_3)$ and that is the function we want to obtain here. To get the implicit equation we may use the "kernel" in `Macaulay 2`.

In the next theorem we will be thinking of $c = n - d$.

Theorem 5.1 *([3]) Let ω be a generic vector in $\mathbb{R}^n$. A monomial prime $< x_{J_1}, ...x_{J_c} >$ is associated to the initial monomial ideal $in_\omega(I)$ if and only if $\mathcal{J}(I)$ meets the cone $\omega + \mathbb{R}_{\geq 0}\{e_{J_1}, ..., e_{J_c}\}$. The number of intersection points, counted appropriately, equals the multiplicity of this prime in $in_\omega(I)$.*

5.3 A simple test case: tropical implicitization of curves

Let $f_1, ..., f_n$ be n rational functions in $d = 1$ unknown t. Let $\alpha_1, ..., \alpha_m \in \mathbb{C} \cup \{\infty\}$ be all the poles and zeros of all the f_i. The case of curves is simple because for univariate polynomials the fundamental theorem of algebra tells us that we can write

$$f_i(t) = \prod_{j=1}^{m} (t - \alpha_j)^{u_{ij}}.$$

Due to the residue theorem we know that the m row vectors $u_i = (u_{i1}, ..., u_{in})$ add up to zero in $\mathbb{R}^n$. The union of their rays equals the tropical curve $\mathcal{T}(I)$.

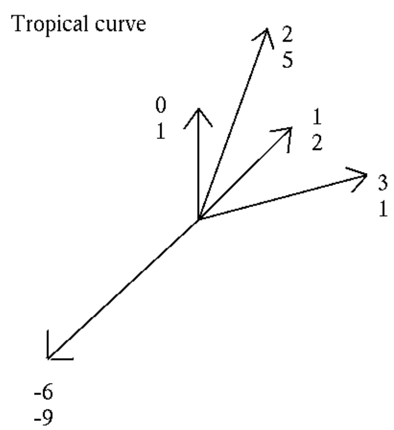
Example. Let us try the following parametrized plane curve

$$\begin{cases} x &= t^2(t-1)^1(t-2)^0(t-3)^3 \\ y &= t^5(t-1)^2(t-2)^1(t-3)^1. \end{cases}$$

We want to obtain the implicit equation of this curve.

The implicit equation has 17 terms:

$$\begin{aligned} g(x, y) &= x^9 + 4x^8 + 494x^7y - 3x^6y^2 + 1978x^6y + \\ &\quad +61,214x^5y^2 + ... + 51,018,336xy^3. \end{aligned}$$



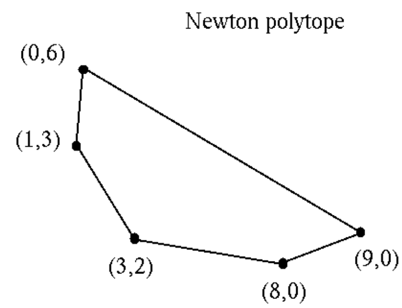


Figure 28: Tropical curve & Newton polytope .

5.4 How about for $d \geq 2$ unknowns?

If $\bigcup_{i=1}^{n}\{f_i = 0\}$ defines a normal crossing divisor with smooth components on some compactification χ of $(\mathbb{C}^*)^d$ then a similar construction works [8].

In order to make this computationally, we will focus on the Newton polytope of the f_i. Suppose the coefficients of f_i are generic relative to fixing the Newton polytope $P_i := New(f_i)$. Choose an $m \times d$-matrix A and column vectors $b_1, ...b_n \in \mathbb{R}^m$ such that for $i = 1, ..., n$;

$$P_i = \{u \in \mathbb{R}^d : Au \geq b_i\}.$$

Example. Plane curves. $(n = 2, d = 1 \Rightarrow m = 2)$ [11]

$$A = \begin{pmatrix} 1 \\ -1 \end{pmatrix},\ b_1 = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix},\ b_2 = \begin{pmatrix} \gamma \\ -\delta \end{pmatrix}$$

The **incidence fan** of $P_1, ..., P_n$ is the coordinate fan in $\mathbb{R}^{n+m}$ with basis $e_1, ..., e_n, E_1, ...E_m$ whose cones are the orthants $\mathbb{R}_{\geq 0}\{e_{i_1}, ..., e_{i_k}, E_{j_1}, ...E_{j_l}\}$ such that the face of $P_{i_1} + ... + P_{i_k}$ indexed by $j_1, ...j_l$ has codimension $\leq l$.

For $l = 0$ take all proper subsets of $\{e_1, ...e_n\}$.

Theorem 5.2 *The tropical variety $\mathcal{T}(I)$ is the image of the incidence fan of $P_1, ..., P_n$ under the linear map*

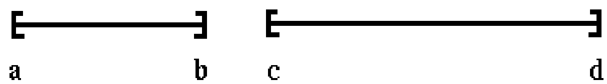
$$\begin{array}{ccc} \mathbb{R}^{n+m} & \longrightarrow & \mathbb{R}^n \\ (y, z) & \mapsto & y + z \cdot B \end{array}$$

where B is the matrix with columns b_i.

If $n = d+1$ and $I =< g >$ is principal, we get a combinatorial rule for constructing the Newton polytope of g from $P_1, ..., P_n$.

5.5 Tropical implicitization of plane curves

`Input:` Two 1-dimensional Newton polytopes .



`Output:` The Newton polygon $Q \subset \mathbb{R}^2$ of the implicit equation $g(x, y) = 0$.

- <u>Case 1</u>

If $a \geq 0$ and $c \geq 0$ then

$$Q = conv\{(0, b),\ (0, a),\ (c, 0),\ (d, 0)\}$$

- <u>Case 2</u>

If $b \leq 0$ and $d \leq 0$ then

$$Q = conv\{(0, -a),\ (0, -b),\ (-d, 0),\ (-c, 0)\}$$

- <u>Case 3</u>

If $a \leq 0, d \geq 0$ and $bc \geq ad$ then

$$Q = conv\{(0, b - a),\ (0, 0),\ (d - c, 0),\ (d, -a)\}$$

- <u>Case 4</u>

If $b \geq 0, c \leq 0$ and $bc \leq ad$ then

$$Q = conv\{(0, b - a),\ (0, 0),\ (d - c, 0),\ (-c, b)\}$$

6 References

References

[1] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, R. Thomas. Computing tropical varieties. `math.AG/0507563`.

[2] D. Cox, J. Little, D. O'Shea. Ideals, varieties, and algorithms. Springer, 1996.

[3] A. Dickenstein, E.M. Feichtner, and B. Sturmfels. Tropical discriminants. `math.AG/0510126`

[4] M. Einsiedler, M. Kapranov, D. Lind. Non-archimedean amoebas and tropical varieties. `math/0408311`

[5] D. Grier, P. Huggins, B. Sturmfels, J. Yu. The hyperdeterminant and triangulations of the 4-cube. `math.CO/0602149`

[6] I.M. Gel'fand, M.M. Kapranov, A.V.Zelevinsky. Discriminants, resultants, and multidimensional determinants. Birkhäuser, Boston, MA. 1994.

[7] B. Poonen, F. Rodriguez Villegas. Lattice Polygons and the Number 12. The American Mathematical Monthly, Vol. 107, No.3. (Mar., 2000), pp. 238-250

[8] P. Hacking, S.Keel, J. Tevelev. Preprint 06.

[9] B. Sturmfels. Solving systems of polynomial equations. Conference Board of the Mathematical Sciences. Regional Conference Series in Mathematics. No. 97.

[10] D. Speyer, B. Sturmfels, The tropical Grassmanian, Adv. Geom. 4 (2004), 389-411. MR 2071813

[11] B. Sturmfels, J. Tevelev, J. Yu. The Newton Polytope of the Implicit Equation. `math.CO/0607368`