Periods for the Fundamental Group

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Course description

Let X be a non singular complex algebraic variety. By Grothendieck, its complex cohomology can be described as (a) singular cohomology, with complex coefficients; (b) hypercohomology of the algebraic de Rham complex Ω_X^*; for X affine: the cohomology of the complex of algebraic differential forms on X. The description (a) gives a rational structure: use rational coefficients. If X is defined over $k \subset \mathbb{C}$, the description (b) gives a k-structure: use forms defined over k. The period matrix is the change of basis matrix, from a rational basis for the $\mathbb{Q}$-structure (a), to a k-basis for the k-structgure (b). Basic example: $k = \mathbb{Q}$, X the multiplicative group $\mathbb{G}_m$. Here, singular H_1 (dual to H^1) is generated by a loop around 0, while de Rham H^1 is generated by $\frac{dz}{z}$. The period matrix is one by one; it is $2\pi i$.

Fix a base point o. Algebraic geometry has few tools to understand $\pi_1(X, o)$ itself. If we make π_1 abelian, we obtain H_1, which has a de Rham description, periods, $\ldots$. The story is almost as good for the group algebra $\mathbb{Q}[\pi_1(X, o)]$, divided by a power of the augmentation ideal I, for instance because this quotient has a description as some relative homology group in X^N. While periods in cohomology have mainly been considered for projective X, $\mathbb{Q}[\pi_1]/I^{N+1}$ is interesting for X as simple as $\mathbb{P}^1 - \{0, 1, \infty\}$. For any X, I/I^2 is H_1, hence I^N/I^{N+1} is a quotient of $\overset{N}{\otimes} H$. For $X = \mathbb{P}^1 - \{0, 1, \infty\}$, the interest lies in the extensions.

The course will explain how for $X = \mathbb{P}^1 - \{0, 1, \infty\}$, the periods of $\mathbb{Q}[\pi_1]/I^{N+1}$ are encoded in the multi-zeta values: the values of

$$\zeta(s_1, \ldots, s_r) = \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}$$

for s_i integers ≥ 1. This for a suitable (tangential) base point.

If we consider X defined over $\mathbb{Q}$, with "good reduction" $\mod p$, the de Rham analog of $\mathbb{Q}[\pi_1]/I^{N+1}$, tensored with $\mathbb{Q}_p$, depends only on the reduction $\mod p$ of X, and is acted upon by its Frobenius endomorphism. For $X = \mathbb{P}^1 - \{0, 1, \infty\}$, this de Rham analog is the quotient of the algebra of non commutative formal power series $\mathbb{Q} \ll e_0, e_1 \gg$ by the part of degree $\geq N + 1$. For the same base point as previously, the Frobenius action on $\mathbb{Q}_p \ll e_0, e_1 \gg$ is of the form

$$
\begin{aligned}
e_1 &\rightarrow p\, e_0 \\
e_1 &\rightarrow g^{-1}.p\, e_1.g,
\end{aligned}
$$

where in g the coefficient of 1 is 1 and that of e_1^n $(n > 0)$ is 0.

Define $\zeta^{(p)}(s_1, \ldots, s_r)$ to be the coefficient in g of $e_0^{s_1-1}e_1e_0^{s_2-1}e_1 \ldots e_0^{s_r-1}e_1$. We will explain why the $\zeta^{(p)}(s_1, \ldots, s_r) \in \mathbb{Q}_p$ should satisfy the same polynomial identities (with rational coefficients) as the $\zeta(s_1, \ldots, s_r) \in \mathbb{R}$, plus the analog of "$2\pi i = 0$" (vanishing of the $\zeta^{(p)}(2n)$).