Buzzard's Group Presentation March 14, 2001

Let p = 2. Consider the modular curve $X_0(2) = (\Gamma_0(2) \setminus \mathcal{H})^*$ which parametrizes pairs (E, C) where C is a subgroup of order 2. As an affine curve, $X_0(2)$ is given by $xy = 2^{12}$. And we define

$$f := 1/x = \frac{\Delta(q^2)}{\Delta(q)} = q \prod_{n=1}^{\infty} (1+q^n)^{24}$$

Since $\Delta(q)$, $\Delta(q^2)$ are weight 12 level 2 modular forms, f is a modular function of weight 0 level 2. We have a cannonical map of degree 3 between $X_0(2)$ and $X_0(1)$, which maps the two cusps $0, \infty$ of $X_0(2)$ to the one cusp ∞ of $X_0(1)$ and preserves q-expansions. (Picture was drawn here during the talk.) Since $\Delta(q)$ has a simple zero at ∞ and a double zero at 0 on $X_0(2)$ and $\Delta(q^2)$ has a double zero at ∞ and a single zero at 0 on $X_0(2)$, f has a simple zero at ∞ and a simple pole at 0. Therefore,

$$f: X_0(2) \xrightarrow{\sim} \mathbb{P}^1$$

Now if we consider $X_0(2)$ as a rigid analytic space, the collection $\{1, f, f^2, f^3, ...\}$ forms a basis of the space of functions on $\{x \in X_0(2) : |f(x)| \leq 1\}$ such that $\sum_{n=0}^{\infty} a_n T^n$, $|a_n|_2 \to 0$ as $n \to \infty$, and where |T| < 1. The reduction of $X_0(2)_{\mathbb{Q}_2}$ to $X_0(2)_{\mathbb{F}_2}$ takes an 2-adic annulus of supersingular points $2^{-12} > |x| > 1$ to the singular point of xy = 0. (Picture could be drawn here.)

If we write $f(q) = \sum_{n=1}^{\infty} a_n q^n$, then our goal is to compute what U_2 looks like on $\{1, f, f^2, f^3, \ldots\}$. If we let $\alpha = f(\sqrt{q})$ and $\beta = f(-\sqrt{q})$, then we have

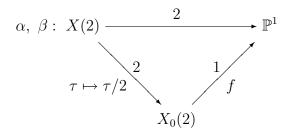
$$\alpha + \beta = \sum a_n q^{n/2} + \sum (-1)^n a_n q^{n/2}$$

= $2 \sum a_n q^{n/2} + 2 \sum a_{2n} q^n$
= $2U_2 f$

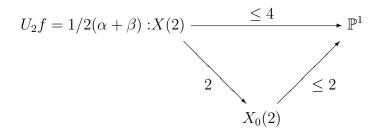
Therefore $U_2 f = \frac{1}{2}(\alpha + \beta)$, and $U_2 f : X_0(2) \longrightarrow \mathbb{P}^1_{\overline{\mathbb{Q}}_2}$ is a map of degree 2.

If f(q) is a modular form for Γ , then $f(\delta q)$ is a modular form for $\delta^{-1}\Gamma\delta$. So $f(\sqrt{q}) = f(\tau/2) = f(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tau)$ is a modular form on $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}^{-1}\Gamma_0(2)\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \supseteq \Gamma(2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 2 \right\}$

Note that we know the degree of α , β are 2, and looking at where our maps factor through we have



Thus $\deg(U_2 f) \leq 4$ on X(2) and we have the following diagram



So $U_2 f$ will factor through $X_0(2)$. Now the question becomes how far out do we need to look for f expansions?

By the above discussion we see that $\deg(U_2 f) = 2$ on $X_0(2)$, so

$$U_2 f = \frac{-+ f_2 - f_2}{-+ f_2 - f_2}$$

and we need to figure out what should go in the blanks. Since $f: 0 \to \infty$ and $f: \infty \to 0$, $U_2 f = - + -f + -f^2$. By a computer calculation we see that $U_2 f = 24f + 2084f^2$.

The matrix for $2U_2$ with respect to $\{1, f, f^2, f^3, ...\}$ will be denoted by (c_{ij}) , where $2U_2f^j = \sum_i c_{ij}f^i$. And this matrix actually begins

(2)	0	0	\longrightarrow
0	48	2	
0	2^{12}	2304	
0	0	÷	
$\langle \uparrow$	\downarrow	÷)

So we are interested in a generating function. Let $g = \sum b_n q^n$, then $2U_2g = 2 \sum b_{2n}q^n = g(\sqrt{q}) + g(-\sqrt{q})$. So if $g = f^j$, $2u_2f^j = \alpha^j + \beta^j = \sum c_{ij}f^i$. Now

$$\sum (\alpha y)^{j} + (\beta y)^{j} = \sum c_{ij} f^{i} y^{j}$$
$$= \frac{1}{1 - \alpha y} + \frac{1}{1 - \beta y}$$
$$= \frac{2 - (\alpha + \beta)y}{1 - (\alpha + \beta)y + \alpha \beta y^{2}}$$

In fact

$$\begin{aligned} \alpha \beta &= \sqrt{q} \prod_{n} \left(1 + (\sqrt{q})^{n} \right)^{24} \cdot -\sqrt{q} \prod_{n} \left(1 + (-\sqrt{q})^{n} \right)^{24} \\ &= -q \prod_{n \text{ even}} \left((1+q^{n})^{2} \right)^{24} \cdot \prod_{n \text{ odd}} (1-q^{n})^{24} \\ &= -q \prod_{n} \frac{\left((1+q^{n})^{2} \right)^{24} \left(1-q^{n} \right)^{24}}{(1-q^{2n})^{24}} \\ &= -q \prod_{n} \left(1+q^{n} \right)^{24} \\ &= -f \end{aligned}$$

So we have a generating function

$$\frac{2 - (48f + 2^{12}f^2)y}{1 - (48f + 2^{12}f^2)y - fy^2}$$

Can you figure out the c_{ij} from this generating function?

In order to make our argument rigorous, we need to show that $2U_2$ is a compact operator on the ρ -overconvergent modular forms with $|\rho|_2 < 1$. We know that if $M = (m_{ij})$ is an infinite matrix giving an operator, then M is compact if and only if for $\gamma_i = \sup_j |m_{ij}|_2$, we have that $\gamma_i \to 0$ as $i \to \infty$. The problem we face is that $c_{i,2i} = 2$, so we do not have a compact operator for the basis $\{1, f, f^2, f^3, \dots\}$.

Since we do not have a compact operator on the space of all 2-adic modular forms, we must restrict to ρ -overconvergent modular forms. Pick $\omega \in \overline{\mathbb{Q}}_2$ such that $|\omega|_2 < 1$. We now adjust our basis: $\{1, (\omega f), (\omega f)^2, \ldots\}$ and look at $2U_2$ as a matrix with respect to this new basis

$$2U_2(\omega f)^i = \omega^j \sum_i c_{ij} f^i$$
$$= \sum_i c_{ij} \omega^{j-i} (\omega f)^i$$

So in terms of the new basis $2U_2 = (d_{ij}) = (c_{ij}\omega^{j-i})$. Now letting $\omega = 2^l$ with l a positive rational and recalling the generating function for (c_{ij}) , we have

$$\sum d_{ij}f^{i}y^{j} = \sum c_{ij}\omega^{j-i}(\omega y)^{j}$$

$$= \frac{2 - (2^{4} \cdot 3f + 2^{12}f^{2}\omega^{-1})y}{1 - (2^{4} \cdot 3f + 2^{12}f^{2}\omega^{-1})y - \omega fy^{2}}$$

$$= \frac{2 - (2^{4} \cdot 3f + 2^{12-l}f^{2})y}{1 - (2^{4} \cdot 3f + 2^{12-l}f^{2})y - 2^{l}fy^{2}}$$

which has a power series in $\mathcal{O}_{\overline{\mathbb{Q}}_2}$. Thus $|d_{ij}|_2 < |2^{il}|_2 = 2^{-il} \to 0$ as $i \to 0$. So $2U_2$ is compact on $\{1, (\omega f), (\omega f)^2, \ldots\}$ as desired.

We have been interested in computing (c_{ij}) and a result of Frank Calegari shows this matrix is computable for U_2 .

Theorem. $c_{ij} = \frac{2^{8i-4j}3j(i+j-1)!}{(2i-j)!(2j-i)!}$ where $c_{ij} = 0$ whenever not well-defined.

Proof. There is a recurrence relation

$$F_{1} = 2U_{2}f = \alpha + \beta = 48f + 4096f^{2}$$

$$F_{n} = 2U_{2}f^{n} = \alpha^{n} + \beta^{n}$$

$$F_{n+1} = 2U_{2}f^{n+1} = \alpha^{n+1} + \beta^{n+1} = (\alpha + \beta)(\alpha^{n} + \beta^{n}) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$$

then

$$F_{n+1} = 2U_2 f \cdot 2U_2 f^n + f(2U_2)(f^{n-1})$$

= $(48f + 4096f^2)F_n + fF_{n-1}$

so $c_{i+1,j+1} = 48c_{i,j} + 4096c_{i-1,j} + c_{i,j-1}$ and the rest follows by an ugly calculation.

Ideally we would like to have a closed form, which we hope to derive from

$$\frac{2 - (\alpha + \beta)y}{1 - (\alpha + \beta)y + \alpha\beta y^2}$$

but what we have been able to compute is that

$$c_{ij} = 2^{8i-4j} 3^{2j-1} \left(\begin{pmatrix} j \\ 2j-i \end{pmatrix} + j \sum_{k=1}^{\left[\frac{2j-i}{3}\right]} \frac{3^{-3k}}{k} \begin{pmatrix} j-(k+1) \\ j-2k \end{pmatrix} \begin{pmatrix} j-2k \\ 2j-i-3k \end{pmatrix} \right)$$

but we have been unable to reduce this to Frank's result.

There are many related open and fun problems. We can try the same calculations with $E_2(q) =$ weight 2, level 2 Eisenstein series, then $E_2(q)$, $E_2(q^2)$ are overconvergent modular forms of weight 2 and level 1. Let $g = \frac{E_2(q)/E_2(q^2) - 1}{24}$ (which is overconvergent of weight 0 and level 2) try to compute the matrix in terms of the parameters

$$U_{2}(g) = 0$$

$$U_{2}(g^{2}) = \sum \left((2i+1)3^{i-2}2^{3(j-1)}(-1)^{i+1} \right) g^{i}$$

$$U_{2}(g^{3}) = 0$$

$$U_{2}(g^{4}) = \pm \left(U_{2}(g^{2}) \right)^{2}$$

There are many interesting problems for $p = 3, 5, 7, 13, \ldots$ What are nice parameters ... try to find nice formulae.