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Elements of Grassmann analysis

9.1 Introduction

One may pose the question, “Why Grassmann analysis?” The answer is that many objects possess intrinsic parity, which the number line \mathbf{R} does not portray. Parity describes the behavior of a product under exchange of its two factors. The so-called Koszul’s parity rule states:

“Whenever you interchange two factors of parity 1, you get a minus sign.”

Formally the rule defines graded commutative products

$$AB = (-1)^{\tilde{A}\tilde{B}}BA \quad (9.1)$$

written $[A, B] = 0$, where $\tilde{A} \in \{0, 1\}$ denotes the parity of A . Objects with parity zero are called *even*, and objects with parity one *odd*. The rule also defines graded anticommutative products. For instance,

$$A \wedge B = -(-1)^{\tilde{A}\tilde{B}}B \wedge A \quad (9.2)$$

written $\{A, B\} = 0$.

A graded commutative product $[A, B]$ can be either a commutator $[A, B]_-$, or an anticommutator $[A, B]_+$. A graded anticommutative product $\{A, B\}$ can be either an anticommutator $\{A, B\}_+$, or a commutator $\{A, B\}_-$. Most often, the context makes it unnecessary to use the $+$ and $-$ signs.

There are no (anti)commutative rules for vectors and matrices. Parity is assigned to such objects in the following way:

- The parity of a vector is determined by its behavior under multipli-

cation with a scalar z :

$$zX = (-1)^{\tilde{z}\tilde{X}} Xz \quad (9.3)$$

- A matrix is even if it preserves the parity of graded vectors and odd if it inverts the parity.

Vectors and matrices do not necessarily have well-defined parity, but they can always be decomposed into a sum of even and odd parts.

One may also ask, “Why Grassmann analysis in Quantum Field Theory?” It is often, but erroneously stated that, since Fermi fields’ anticommutators vanish in the limit $\hbar = 0$, the “classical” limits of Fermi fields take their values in a Grassmann algebra. The flaw in this argument is that the canonical anticommutation relations do not depend on \hbar . Indeed, the canonical quantization rules are

$$[\Phi(x), \Pi(y)]_- = i\hbar\delta(x-y) \quad \text{for a bosonic system} \quad (9.4)$$

$$\{\psi(x), \pi(y)\}_+ = i\hbar\delta(x-y) \quad \text{for a fermionic system.} \quad (9.5)$$

Given the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(-p_\mu\gamma^\mu - mc)\psi = i\hbar\bar{\psi}\gamma^\mu\partial_\mu\psi - mc\bar{\psi}\psi, \quad (9.6)$$

the conjugate momentum $\pi(x) = \delta\mathcal{L}/\delta\dot{\psi}$ is proportional to \hbar . The net result is that the anticommutator is independent of \hbar .

The usefulness of Grassmann analysis in physics is convincingly stated in papers† by F.A. Berezin and M.S. Marinov. Functional integral representation of matrix elements of Fermi operators (operators with values in Clifford algebra) are integrals over functionals of Grassmann variables.

One last question arises: “How advanced is the theory of functional integrals over Grassmann variables?” The short answer is that the theory is complete for integrals over finite dimensional graded manifolds, but its application to infinite dimensional graded manifolds is still being investigated.

We refer to [#95] for the different definitions of graded manifolds, supermanifolds, supervarieties, superspace, and sliced manifolds proposed by various people. Appendix IIIC includes a compendium of Grassmann

† Selected quotes can be found in [#95]

analysis, extracted from the Master's Thesis[‡] of Maria E. Bell. Here we consider simply superfunctions F on $\mathbf{R}^n|\nu$, that is, functions of n real variables $\{x^a\}$ and ν Grassmann variables $\{\xi^\alpha\}$:

$$F(x, \xi) = \sum_{p=0}^{\nu} \frac{1}{p!} f_{\alpha_1 \dots \alpha_p}(x) \xi^{\alpha_1} \dots \xi^{\alpha_p} \quad (9.7)$$

where the functions $f_{\alpha_1 \dots \alpha_p}$ are smooth functions on \mathbf{R}^n , antisymmetric in the indices $\alpha_1, \dots, \alpha_n$.

9.2 Berezin Integration

[‡] Bell, Maria E. (2002). *Introduction to Supersymmetry*. Unpublished Master's Thesis, University of Texas, Austin.