3

Selected Examples

3.1 Wiener measure and brownian paths (Discretizing a path integral)

Let $\boldsymbol{X}$ be a space of continuous pointed paths x over a time interval $\mathbb{T} = [t_a, t_b]$

$$\begin{cases} x : \mathbb{T} \to \mathbb{R} & \text{by} & t \mapsto x(t) \\ x(t_0) = a & \text{for every} & x \in \boldsymbol{X}, t_0 \in \mathbb{T} \end{cases} \tag{3.1}$$

The dual $\boldsymbol{X}'$ of $\boldsymbol{X}$ is the space of bounded measures x' on $\mathbb{R}$

$$\int_{\boldsymbol{T}} dx'(t) < \infty \Leftrightarrow x' \in \boldsymbol{X}'. \tag{3.2}$$

Let us discretize the time interval $\mathbb{T}$ into n variables

$$t_a \leq t_1 \leq \cdots \leq t_n \leq t_b. \tag{3.3}$$

Let $\boldsymbol{Y}$ be the Wiener differential space consisting of the differences of two consecutive values of x on the discretized time interval

$$y^j = x\left(t_{j+1}\right) - x\left(t_j\right) = \left\langle \delta_{t_{j+1}} - \delta_{t_j}, x \right\rangle, \text{ with } t_0 = t_1. \tag{3.4}$$

Therefore the discretizing map $L : \boldsymbol{X} \to \boldsymbol{Y}$ is a projection from the infinite dimensional space $\boldsymbol{X}$ onto the n-dimensional space $\boldsymbol{Y}$.

Let $\gamma_{\mathbb{X}}$ be the Wiener Gaussian defined by its Fourier transforms $\mathcal{F}\gamma_{\mathbb{X}}$, i.e. by the variance

$$W_{\mathbb{X}'}(x') = \int_{\boldsymbol{T}} dx'(t) \int_{\boldsymbol{T}} dx'(s) \inf(t - t_a, s - t_a) \tag{3.5}$$

where $\inf(t, s)$ is the smaller of t and s:

$$\inf(t - t_a, s - t_a) = \theta(t - s)\,(s - t_a) + \theta(s - t)\,(t - t_a)\,; \tag{3.6}$$

the step function θ is equal 1 for positive arguments, equal to 0 for negative arguments, and discontinuous at the origin. The transpose $\tilde{L}$ of L is defined by (2.48**??**), i.e.

$$\begin{aligned}
\langle \tilde{L}y', x\rangle_{\mathbb{X}} &= \langle y', Lx\rangle_{\boldsymbol{Y}} = \sum_j y'_j y^j \\
&= \sum_j y'_j \left\langle \delta_{t_{j+1}} - \delta_{t_j}, x\right\rangle \\
&= \sum_j \left\langle y'_j \big(\delta_{t_{j+1}} - \delta_{t_j}\big), x\right\rangle
\end{aligned}$$

hence

$$\tilde{L}y' = \sum_j y'_j \big(\delta_{t_{j+1}} - \delta_{t_j}\big) \tag{3.7}$$

and

$$\begin{aligned}
W_{\boldsymbol{Y}'}(y') &= \Big(W_{\mathbb{X}'} \circ \tilde{L}\Big)(y') = W_{\mathbb{X}'}\Big(\tilde{L}y'\Big) \\
&= \sum_{j,k} y'_j y'_k \big(\inf(t_{j+1}, t_{k+1}) - \inf(t_{j+1}, t_k) \\
&\qquad - \inf(t_j, t_{k+1}) + \inf(t_j, t_k)\big).
\end{aligned}$$

The terms $j \neq k$ do not contribute to $W_{\boldsymbol{Y}'}(y')$. Hence

$$W_{\boldsymbol{Y}'}(y') = \sum_j \left(y'_j\right)^2 (t_{j+1} - t_j)\,. \tag{3.8}$$

Choosing $\boldsymbol{Y}$ to be a Wiener differential space rather than a naively discretized space defined by $\{x(t_j)\}$ diagonalizes the variance $W_{\boldsymbol{Y}'}(y')$. The Gaussian $\gamma_{\boldsymbol{Y}}$ on $\boldsymbol{Y}$ defined by the Fourier transforms

$$(\mathcal{F}\gamma_{\boldsymbol{Y}})(y') = \exp\left(-s\pi \sum_{i,j} \delta^{i,j} y'_i y'_j \,(t_{j+1} - t_j)\right) \tag{3.9}$$

is

$$d\gamma_{\boldsymbol{Y}}(y) = dy^1 \cdots dy^n \frac{1}{\prod_j^n \left(s\,(t_{j+1} - t_j)\right)^{1/2}} \;\exp\left(-\frac{\pi}{s} \sum_{i,j} \frac{\delta_{ij} y^i y^j}{t_{j+1} - t_j}\right) \tag{3.10}$$

set

$$\Delta t_j := t_{j+1} - t_j,$$

$$\Delta x^j := (\Delta x)^j := x(t_{j+1}) - x(t_j) = y^j ,$$

$$x^j := x(t_j) ,$$

then

$$d\gamma_{\boldsymbol{Y}}(\Delta x) = dx^1 \cdots dx^n \frac{1}{\prod_{j=1}^n (s\Delta t_j)^{1/2}} \; \exp\left(-\frac{\pi}{s} \sum_j \frac{\left(\Delta x^j\right)^2}{\Delta t_j} \right) . \tag{3.11}$$

When $s = 1$, the Gaussian $\gamma_{\boldsymbol{Y}}$ defines the distribution of a Brownian path; The Gaussian $\gamma_{\mathbb{X}}$ of covariance $\inf(t - t_a, s - t_a)$ is the Wiener measure.

Deriving the distribution of Brownian paths from the Wiener measure is a particular case of a general formula. Let $F : \boldsymbol{X} \to \mathbb{R}$ be a functional on $\boldsymbol{X}$ which can be decomposed into two maps $F = f \circ L$ where

$$L : \; \boldsymbol{X} \to \boldsymbol{Y} \text{ linearly}$$

$$f : \boldsymbol{Y} \to \mathbb{R} \text{ integrable with respect to } \gamma_{\boldsymbol{Y}}$$

then

$$\int_{\mathbb{X}} D\gamma_{\mathbb{X}} \; F(x) = \int_{\boldsymbol{Y}} D\gamma_{\boldsymbol{Y}}(y) \; f(y), \tag{3.12}$$

where the Gaussians $\gamma_{\mathbb{X}}$ and $\gamma_{\boldsymbol{Y}}$ are characterized by the quadratic form $W_{\mathbb{X}'}$, and $W_{\boldsymbol{Y}'}$ such that

$$W_{\boldsymbol{Y}'} = W_{\mathbb{X}'} \circ \tilde{L}. \tag{3.13}$$

3.2 Canonical Gaussians in $\boldsymbol{L^2}$ and $\boldsymbol{L^{2,1}}$

Wiener Gaussians on spaces of continuous paths serve probabilists very effectively, but physicists who work with kinetic energy use $L^{2,1}$ spaces, i.e. Sobolev spaces of square integrable functions on $\mathbb{T}$ whose first derivatives (in the sense of distribution) are square integrable. Let $\mathcal{H}$ be the Hilbert space of real square integrable functions h on $\mathbb{T}$ with norm:

$$||h||^2_{\mathcal{H}} = \int_{\boldsymbol{T}} dt \; g_{ij} h^i(t) \; h^j(t). \tag{3.14}$$

Let $L^{2,1}_- \subset L^{2,1}$ be the subspace of functions on $\mathbb{T} = [t_a, t_b]$ vanishing at t_a and $L^{2,1}_+ \subset L^{2,1}$ be subspace of functions vanishing at t_b, with the $L^{2,1}$ norm

$$||y||_{L^{2,1}} := \int_{\boldsymbol{T}} dt\ g_{ij}\dot{y}^i(t)\ \dot{y}^j(t), \qquad \dot{y}(t) := \frac{dy(t)}{dt}. \tag{3.15}$$

Paths vanishing at t_a are used in diffusion problems, paths vanishing at t_b are used in solutions $\psi(t_b, \mathbf{x})$ of the Schrödinger equation for the probability amplitude of finding the system at $\mathbf{x}$ at time t_b.

Spaces of pointed paths (spaces of paths taking the same value at a given time) are particularly useful because they are contractible [See Section (7.1**??**)]. One of the linearity conditions for a space of pointed paths is that the common value of the paths be zero; indeed, for

$$x_{(i)} \in \boldsymbol{X}_a, \text{ and } x_{(i)}(t_a) = \mathbf{x}_a \text{ for all } x_{(i)} \in \boldsymbol{X}_a,$$

$$x_{(i)} + x_{(j)} \in \boldsymbol{X}_a \text{ only if } \mathbf{x}_a = 0.$$

Let $\boldsymbol{Y}_-$ and $\boldsymbol{Y}_+$ be the spaces of continuous paths vanishing at t_a and t_b, respectively. The primitive mapping $P_- : \mathcal{H} \to L^{2,1}_-$ by

$$P_- : h(t) \to y(t) = \int_{t_a}^{t} ds\ h(s) = \int_{\boldsymbol{T}} ds\ \theta(t-s)\ h(s), \tag{3.16}$$

the primitive mapping $P_+ :\ \mathcal{H} \to L^{2,1}_+$ by

$$P_+ : h(t) \to y(t) = -\int_{t}^{t_b} ds\ h(s) = -\int_{\boldsymbol{T}} ds\ \theta(s-t)\ h(s), \tag{3.17}$$

the inclusion mapping $i_\mp : L^{2,1}_\mp \to \boldsymbol{Y}_\mp$ by

$$i_\mp : y \in L^{2,1}_\mp \mapsto y \in \boldsymbol{Y}_\mp, \tag{3.18}$$

not a trivial mapping, as we shall see shortly. The composition

$$P^W_\mp := i_\mp \circ P_\mp :\ \mathcal{H} \to \boldsymbol{Y}_\mp. \tag{3.19}$$

The quadratic form $W_{\mathcal{H}'}$ defining a canonical Gaussian $\gamma_{\mathcal{H}}$ on a Hilbert space $\mathcal{H}$ is equal to the norm of the dual $\mathcal{H}'$ of $\mathcal{H}$. We shall show that

(i) $P^W_\mp$ maps the canonical Gaussian on $\mathcal{H}$ into the Wiener Gaussian on $\boldsymbol{Y}_\mp$

Fig. 3.1. Canonical gaussians on L^2 and $L^{2,1}$.

(ii) the inclusion $i_\mp$ maps the canonical Gaussian on $L^{2,1}_\mp$ into the Wiener Gaussian on $\boldsymbol{Y}_\mp$.

The proofs are applications of (2.34**??**), namely composition of quadratic forms induced by linear mappings: if $L: \ \boldsymbol{X} \to \boldsymbol{Y}$ linearly, then

$$W_{\boldsymbol{Y}'} = W_{\boldsymbol{X}'} \circ \tilde{L}, \quad \text{where} \quad \left\langle \tilde{L}y', x \right\rangle = \langle y', Lx \rangle \, .$$

(i) The transpose $\tilde{P}^W_-$ is defined on $\boldsymbol{Y}'$ by

$$\left\langle \tilde{P}^W_- y', h \right\rangle_{\mathcal{H}} = \langle y', P^W_- h \rangle_{\boldsymbol{Y}_-} = \int_{\boldsymbol{T}} dy'_j(t) \int_{\boldsymbol{T}} ds \ \theta(t-s) \ h^j(s), \quad (3.20)$$

therefore

$$\left(\tilde{P}_{-}^{W} y'\right)_j (s) = \int_{\boldsymbol{T}} dy'_j(t)\ \theta(t-s), \tag{3.21}$$

and the quadratic form $W_{\mathcal{H}'}$ is equal to the norm on $\mathcal{H}'$ defined by the norm (2.45**??**) on $\mathcal{H}$

$$\begin{aligned}\left(W_{\mathcal{H}'} \circ \tilde{P}_{-}^{W}\right)(y') &= \int_{\boldsymbol{T}} ds\ g^{ij} \int_{\boldsymbol{T}} \theta(t-s)\ dy'_i(t) \int_{\boldsymbol{T}} \theta(t'-s)\ dy'_j(t') \\ &= g^{ij} \int_{\boldsymbol{T}} dy'_i(t) \int_{\boldsymbol{T}} dy'_j(t')\ \inf(t-t_a, t'-t_a) \quad (3.22) \\ &= g^{ij} \int_{t_a}^{\inf(t,t')} ds \int_{\boldsymbol{T}} dy'_i(t) \int_{\boldsymbol{T}} dy'_j(t)\ .\end{aligned}$$

$\square$

(ii) The transpose $\tilde{i}$ on $\boldsymbol{Y}'_{-}$ is defined by

$$\left\langle \tilde{i}y', y \right\rangle_{L_{-}^{2,1}} = \langle y', iy \rangle_{\boldsymbol{Y}_{-}}\ , \tag{3.23}$$

i.e. explicitly, given the dualities (3.15) and (3.2)

$$\int_{\boldsymbol{T}} dt \frac{d}{dt} (\tilde{i}y')_j(t) \cdot \frac{d}{dt} y^j(t) = \int_{\boldsymbol{T}} dy'_j(t)\ y^j(t), \tag{3.24}$$

hence

$$\frac{d}{dr} \left(\tilde{i}y'\right)_j (r) = \int_{\boldsymbol{T}} \theta(t-r)\ dy'_j(t), \tag{3.25}$$

and

$$\begin{aligned}\left(W_{L'^{2,1}_{-}} \circ \tilde{i}\right)(y') &= \int_{\boldsymbol{T}} dr\ g^{ij} \frac{d}{dr} \left(\tilde{i}y'\right)_i (r) \cdot \frac{d}{dr} \left(\tilde{i}y'\right)_j (r) \\ &= g^{ij} \int_{\boldsymbol{T}} dy'_i(t) \int_{\boldsymbol{T}} dy'_j(t')\ \inf(t-t_a, t'-t_a)\,. \quad (3.26)\end{aligned}$$

$\square$

The mappings $P_{\mp}$ and $i_{\mp}$ defined on Fig. 3.1 have been generalized by A. Maheshwari [1] and used in [2] for various versions of the Cameron-Martin formula and for Fredholm determinants.

3.3 The Forced Harmonic Oscillator

Up to this point we have exploited properties of Gaussian γ_Q on $\boldsymbol{X}$ defined by quadratic forms W on the dual $\boldsymbol{X}'$ of $\boldsymbol{X}$ but we have not used the quadratic form Q on $\boldsymbol{X}$ inverse of W in the following sense; set

$$\begin{aligned} W(x') &=: \langle x', Gx' \rangle \\ Q(x) &=: \langle Dx, x \rangle . \end{aligned} \tag{3.27}$$

W and Q are said to be inverse of each other if

$$DG = \mathbb{1}_{\mathbb{X}'} \ , \quad GD = \mathbb{1}_{\mathbb{X}}. \tag{3.28}$$

The gaussian volume element $d\gamma_{s,Q}$ can be expressed in terms of the quadratic form Q:

$$\mathcal{D}_{s,Q}(x) \ \exp\left(-\frac{\pi}{s} \ Q(x)\right) \stackrel{\int}{=} d\gamma_{s,Q}; \tag{3.29}$$

this is a qualified equality, meaning that both expressions are defined by the same integral equation (2.25**??**). In this section, we use gaussians defined by Q with $s = i$ and $D_{s,Q}$ is abbreviated to D_Q.

The first path integral proposed by Feynman was

$$\langle b, t_b | a, t_a \rangle = \int_{\mathbb{X}_{a,b}} Dx \ \exp\left(\imath S(x)/\hbar\right) \tag{3.30}$$

with

$$S(x) = \int_{t_a}^{t_b} dt \left(\frac{m}{2} \left(\dot{x}(t)\right)^2 - V(x(t)) \right) \tag{3.31}$$

where

- $\langle b, t_b | a, t_a \rangle$ is the probability amplitude that a particle in the state a at time t_a be found at b at time t_b.
- the domain of integration $\boldsymbol{X}_{a,b}$ is the space of paths $x : \ [t_a, t_b] \to \mathbb{R}$ such that $x(t_a) = a$ and $x(t_b) = b$, $x \in \boldsymbol{X}_{a,b}$.

Domain of Integration $\boldsymbol{X}_{a,b}$ and the Normalization of its Volume Element

$\boldsymbol{X}_{a,b}$ is not a vector space† unless $a = 0$ and $b = 0$. Satisfying *only one* of these two vanishing requirements is easy and beneficial:

- easy; indeed choose the origin of the coordinates of $\mathbb{R}$ to be either a or b. The condition $x(t_a) = 0$ is convenient for problems in diffusion, the condition $x(t_b) = 0$ is convenient for problems in quantum mechanics,

† Let $x \in \boldsymbol{X}_{a,b}$ and $y \in \boldsymbol{X}_{a,b}$, then $(x+y) \in \boldsymbol{X}_{a,b}$ only if $(x+y)(t_a) = a$ and $(x+y)(t_b) = b$.

- beneficial; a space of pointed paths is contractible (see section 7.1**??**) and can be mapped into a space of paths on $\mathbb{R}^D$.

Therefore we rewrite (3.30) as an integral over $\boldsymbol{X}_b$, the space of paths vanishing at t_b; the other requirement is achieved by introducing $\delta\left(x(t_a) - a\right)$:

$$\langle b, t_b | a, t_a \rangle = \int_{\mathbb{X}_b} Dx \;\; \exp\left(\imath S(x)/\hbar\right) \;\; \delta(x(t_a) - a). \tag{3.32}$$

This is a particular case of

$$\langle b, t_b | \phi, t_a \rangle = \int_{\mathbb{X}_b} Dx \;\; \exp\left(\imath S(x)/\hbar\right) \;\; \phi(x(t_a)), \tag{3.33}$$

useful for solving Schrödinger equations, given an initial wave function ϕ.

An affine transformation from $\boldsymbol{X}_{a,b}$ onto $\boldsymbol{X}_{0,0}$ expresses "the first path integral" (3.30, 3.32) as an integral over a Banach space. This affine transformation is known as the background method. We present it here in the simplest case of paths with values in $\mathbb{R}^D$. For more general cases see chapter 4**??** on semiclassical expansions, and for the general case of a map from a space of pointed paths on a riemannian manifold $\mathbb{M}^D$ to a space of pointed paths on $\mathbb{R}^D$ see section 7.1**??**.

Let $x \in \boldsymbol{X}_{0,0}$ and y be a fixed arbitrary path in $\boldsymbol{X}_{a,b}$, possibly a classical path defined by the action functional, but not necessarily so.

$$(y + x) \in \boldsymbol{X}_{a,b} \;\; , \;\; x \in \boldsymbol{X}_{0,0} \tag{3.34}$$

Let $\gamma_{s,Q}$, abbreviated to γ, be the gaussian defined by

$$\int_{\mathbb{X}_b} d\gamma(x) \exp(-2\pi\imath \langle x', x \rangle) := \exp(-\pi s W_b(x')), \tag{3.35}$$

where $\boldsymbol{X}_b$ is the space of paths vanishing at t_b

$$x(t_b) = 0.$$

The gaussian volume defined by (3.35) is normalized to 1:

$$\gamma(\boldsymbol{X}_b) := \int_{\mathbb{X}_b} d\gamma(x) = 1. \tag{3.36}$$

The gaussian volume element on $\boldsymbol{X}_{0,0}$ is readily computed by the linear map

$$L \; : \; \boldsymbol{X}_{b=0} \;\longrightarrow\; \mathbb{R}^D \quad \text{by} \quad x^i \longrightarrow u^i := \left\langle \delta_{t_a}, x^i \right\rangle$$

$$\gamma_{0,0}(\boldsymbol{X}_{0,0}) := \int_{\mathbb{X}_{b=0}} d\gamma(x)\ \delta\big(x^1(t_a)\big)\ \dots\ \delta\big(x^D(t_a)\big) \quad (3.37)$$

$$= \int_{\mathbb{R}} d\gamma^L(u)\ \delta(u^1)\ \dots\ \delta\big(u^D\big) \quad (3.38)$$

where the gaussian γ^L is defined by the variance

$$W^L = W \circ \tilde{L}$$

with

$$\left\langle \left(\tilde{L}u'\right)_i, x^i \right\rangle = u'_i L x^i = \langle u'_i \delta_{t_a}, x^i \rangle .$$

Therefore, using $W(\delta_{t_a}) = \int dt\ \delta_{t_a}(t)\ \int ds\ \delta_{t_a}(s)\ G(t,s)$,

$$W^L(u') = W(u'\delta_{t_a}) = u'_i u'_j G^{ij}(t_a, t_a)$$

and

$$d\gamma^L(u) = du(\det G_{ij}(t_a,t_a)/s)^{-1/2} \exp\Big(-\frac{\pi}{s} u^i u^j G_{ij}(t_a,t_a)\Big)$$

where $G^{ij}G_{jk} = \delta^i_k$.

Finally

$$\gamma_{0,0}(\boldsymbol{X}_{0,0}) = (\det G_{ij}(t_a,t_a)/s)^{-1/2} . \quad (3.39)$$

An affine transformation preserves gaussian normalization, therefore,

$$\gamma_{a,b}(\boldsymbol{X}_{a,b}) = \gamma_{0,0}(\boldsymbol{X}_{0,0}). \quad (3.40)$$

An affine transformation "shifts" a gaussian, and multiplies its Fourier transform by a phase. This property is most simply seen in one dimension: it follows from

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{a}} \exp\Big(-\pi \frac{x^2}{a} - 2\pi\imath \langle x', x\rangle\Big) = \exp(-\pi a x'^2)$$

that

$$\int_{\mathbb{R}} \frac{dx}{\sqrt{a}} \exp\Big(-\pi \frac{(x+l)^2}{a} - 2\pi\imath \langle x', x\rangle\Big)$$
$$= \exp 2\pi\imath \langle x', l\rangle \exp(-\pi a x'^2). \quad (3.41)$$

Under the affine transformation $x \in \boldsymbol{X}_{0,0} \mapsto (y+x) \in \boldsymbol{X}_{a,b}$, $\gamma_{0,0} \mapsto \gamma_{a,b}$; their respective Fourier transforms differ only by a phase, since their gaussian volumes are equal.

$$\int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x) \exp(-2\pi\imath \langle x', x\rangle) = \gamma_{0,0}(\boldsymbol{X}_{0,0}) \exp(-\pi s W_{0,0}(x')) \quad (3.42)$$

$$\int_{\mathbb{X}_{a,b}} d\gamma_{a,b}(x) \exp(-2\pi\imath \langle x', x\rangle)$$
$$= \gamma_{a,b}(\boldsymbol{X}_{a,b}) \exp(2\pi\imath \langle x', y\rangle) \exp(-\pi s W_{0,0}(x')). \quad (3.43)$$

Normalization dictated by Quantum Mechanics†

The first path integral (3.30)

$$\left\langle b \left| \exp\left(-\frac{\imath}{\hbar} H(t_b - t_a)\right) \right| a \right\rangle = \int_{\mathbb{X}_{a,b}} \mathcal{D}x \exp(\imath S(x)/\hbar) \quad (3.44)$$

implies a relationship between the normalization of volume elements in path integrals and the normalization of matrix elements in quantum mechanics, itself dictated by the physical meaning of such matrix elements. Two examples: a free particle, a simple harmonic oscillator. The most common normalization in quantum mechanics is

$$\langle x''|x'\rangle = \delta(x'' - x'), \quad \langle p''|p'\rangle = \delta(p'' - p'); \quad (3.45)$$

it implies [Sakurai] the following normalizations

$$\langle x'|p'\rangle = \frac{1}{\sqrt{h}} \exp 2\pi\imath \langle p', x'\rangle / h \quad (3.46)$$

and

$$|p\rangle = \int_{\mathbb{R}} dx \frac{1}{\sqrt{h}} \exp(2\pi\imath \langle p, x\rangle / h)|x\rangle \quad (3.47)$$

i.e.

$$|p_a = 0\rangle = \frac{1}{\sqrt{h}} \int_{\mathbb{R}} dx|x\rangle. \quad (3.48)$$

The hamiltonian operator H_0 of a free particle of mass m is

$$H_0 := p^2/2m \quad (3.49)$$

and the matrix element

$$\left\langle x_b = 0 \left| \exp\left(-\frac{2\pi\imath}{h} H_0(t_b - t_a)\right) \right| p_a = 0 \right\rangle = \frac{1}{\sqrt{h}}. \quad (3.50)$$

The normalization of this matrix element corresponds to the normalization of the gaussian $\gamma_{Q_0/h}$ on $\boldsymbol{X}_b$

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h}(x) = 1 \quad (3.51)$$

† Contributed by Ryoichi Miyamoto.

which follows from the definition (3.35) of γ_{Q_0}, valid for $s = 1$, and $s = \imath$.

Proof Set $s = \imath$, set

$$Q_0 := \int_{t_a}^{t_b} dt\ m\ \dot{x}^2(t), \tag{3.52}$$

and

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h}(x)\ \delta(x(t_a) = x_a) = \left\langle x_b = 0 \left| \exp\left(-\frac{2\pi\imath}{h} H_0(t_b - t_a)\right) \right| x_a \right\rangle; \tag{3.53}$$

equivalently, $S_0 := \frac{1}{2} Q_0$,

$$\begin{aligned} &\int_{\mathbb{X}_b} \mathcal{D}_{s,Q_0/h}(x) \exp\left(\frac{2\pi\imath}{h} S_0(x)\right)\ \delta(x(t_a) = x_a) \\ &\quad = \left\langle x_b = 0 \left| \exp\left(-\frac{2\pi\imath}{h} H_0(t_b - t_a)\right) \right| x_a \right\rangle. \end{aligned} \tag{3.54}$$

In order to compare the operator and path integrals normalizations, we integrate both sides of (3.53) with respect to x_a.

$$\begin{aligned} \int_{\mathbb{X}_b} d\gamma_{Q_0/h}(x) &= \int dx_a \left\langle x_b = 0 \left| \exp\left(-\frac{2\pi\imath}{h} H_0(t_b - t_a)\right) \right| x_a \right\rangle \\ &= \sqrt{h} \left\langle x_b = 0 \left| \exp\left(-\frac{2\pi\imath}{h} H_0(t_b - t_a)\right) \right| p_a = 0 \right\rangle \text{ , by (3.48)} \\ &= 1 \text{ , by (3.50)} \end{aligned} \tag{3.55}$$

□

The quantum mechanical normalization (3.50) implies the functional path integral normalization (3.51).

The hamiltonian operator $H_0 + H$ for a simple harmonic oscillator is

$$H_0 + H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{3.56}$$

There is a choe of quadratic form for defining the gaussian volume element, Q_0 or $Q_0 + Q_1$,

$$\frac{1}{2}(Q_0 + Q_1) \text{ corresponding respectively to } S_0 + S_1 \text{ and } H_0 + H_1. \tag{3.57}$$

We use first γ_{Q_0}; the starting point (3.53) or (3.54) now reads

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h} \exp\left(\frac{\pi\imath}{h} Q_1(x)\right) \delta(x(t_a) = x_a)$$

$$\equiv \int_{\mathbb{X}_b} \mathcal{D}_{Q_0/h}(x)\ \exp\biggl(\frac{2\pi\imath}{h}\,(S_0+S_1)\,(x)\biggr)\,\delta(x(t_a)=x_a) \quad (3.58)$$
$$= \biggl\langle x_b=0 \biggm| \exp\biggl(-\frac{2\pi\imath}{h}\,(H_0+H_1)\,(t_b-t_a)\biggr)\biggm| x_a\biggr\rangle\,.$$

This matrix element, which can be found, for instance, in [3, (2.5.18)], is used for computing the momentum to position matrix element as follows

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h}\ \exp\frac{\pi\imath}{h}Q_1(x)$$
$$= \sqrt{h}\biggl\langle x_b=0 \biggm| \exp\biggl(-\frac{2\pi\imath}{h}(H_0+H_1)(t_b-t_a)\biggr)\biggm| p_a=0\biggr\rangle$$
$$= \sqrt{h}\int_{\mathbb{R}} dx\,\biggl\langle x_b=0 \biggm|\exp\biggl(-\frac{2\pi\imath}{h}(H_0+H_1)(t_b-t_a)\biggr)\biggm| x\biggr\rangle\biggl\langle x\biggm|p_a=0\biggr\rangle$$

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h}\ \exp\frac{\pi\imath}{h}Q_1(x)$$
$$= \sqrt{h}\int_{\mathbb{R}} dx\ \left(\frac{m\omega}{\imath h\sin(\omega(t_b-t_a))}\right)^{1/2}$$
$$\times\ \exp\biggl(\frac{\pi\imath m\omega}{h\sin(\omega(t_b-t_a))}x^2\cos(\omega(t_b-t_a))\biggr)\,\frac{1}{\sqrt{h}}$$
$$= (\cos\omega(t_b-t_a))^{-1/2}\ . \quad (3.59)$$

On the other hand the l.h.s. of (3.59) is computed in section 4.2?? and found equal to the following ratio of determinants (4.26??)

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h}(x)\ \exp\frac{\pi\imath}{h}Q_1(x)$$
$$\equiv \int_{\mathbb{X}_b} \mathcal{D}_{Q_0}(x)\ \exp\Bigl(-\frac{\pi\imath}{h}\,(Q_0(x)+Q_1(x))\Bigr)$$
$$= \left(\frac{\det Q_0}{\det(Q_0+Q_1)}\right)^{1/2}. \quad (3.60)$$

The ratio of these infinite dimensional determinants is equal to a finite dimensional determinant (appendix IE??)

$$\int_{\mathbb{X}_b} d\gamma_{Q_0/h}(x)\ \exp\frac{\pi\imath}{h}Q_1(x) = (\cos\omega(t_b-t_a))^{-1/2}\,. \quad (3.61)$$

In conclusion the path integral (3.61) is indeed a representation of the matrix element on the r.h.s. of (3.59). This result confirms the normalizations checked in the simpler case of the free particle.

Remark Eq. (3.61) also shows that one would be mistaken in assuming that the gaussian $d\gamma_{Q+Q_0}(x)$ is equal to $d\gamma_{Q_0}(x)\ \exp\frac{\pi\imath}{h}Q_1(x)$. The reason is that

$$d\gamma_{Q_0}(x) \overset{\int}{=} \mathcal{D}_{Q_0}x\ \exp\pi\imath Q_0(x) \tag{3.62}$$

$$d\gamma_{Q_0+Q}(x) \overset{\int}{=} \mathcal{D}_{Q_0+Q}x\ \exp\pi\imath(Q_0+Q_1)(x) \tag{3.63}$$

and

$$\mathcal{D}_{Q_0}/\mathcal{D}_{Q_0+Q_1} \overset{\int}{=} \left|\det Q_0/\det(Q_0+Q_1)\right|^{1/2}. \tag{3.64}$$

□

In the case of the forced harmonic oscillator (next section) it is simpler to work with $\gamma_{Q_0+Q_1}$ than γ_{Q_0}. Given the action

$$S(x) = \frac{1}{2}(Q_0+Q_1) - \lambda\int_{\boldsymbol{T}} dt\ f(t)\ x(t) \tag{3.65}$$

we shall express the integral w.r.t. γ_{Q_0} as an integral w.r.t. γ_{Q+Q_0}

$$\begin{aligned}&\int_{\mathbb{X}_b} d\gamma_{Q_0/h}(x)\ \exp\frac{\pi\imath}{h}Q(x)\ \exp\left(-\frac{2\pi\imath\lambda}{h}\int_{\boldsymbol{T}} dt\ f(t)\ x(t)\right)\\ &= \left|\frac{\det Q_0}{\det Q_0+Q}\right|^{1/2}\int_{\mathbb{X}_b} d\gamma_{(Q_0+Q)/h}(x)\ \exp\left(-\frac{2\pi\imath\lambda}{h}\int_{\boldsymbol{T}} dt\ f(t)\ x(t)\right)\end{aligned} \tag{3.66}$$

and use appendix IE for the explicit value of the ratio of these infinite dimensional determinants, namely

$$\left|\det Q_0/\det Q_0+Q\right|^{1/2} = (\cos\omega(t_b-t_a))^{-1/2}. \tag{3.67}$$

Remark As noted in section 1.1**??**, Wiener showed the key role played by Differential Spaces. Here the kinetic energy Q_0 can be defined on a differential space, Q_0+Q_1 cannot. Therefore, one often needs to begin with Q_0 before introducing more general gaussians.

Choosing a Quadratic Form Q on $\boldsymbol{X}_b$

The integrand, $\exp(iS(x)/\hbar)$, suggests three choices for Q:

- Q can be the kinetic energy.
- Q can be the kinetic energy plus any existing quadratic terms in V.
- Q can be the second term in the functional Taylor expansion of S.

This third option is, in general, the best one because the corresponding gaussian term contains the most information. See for instance section 4.3**??** the anharmonic oscillator. The expansion of S around its value for the classical path (the solution of the Euler-Lagrange equation) is called the semiclassical expansion of the action. It will be exploited in Chapter 4.

The Forced Harmonic Oscillator

The action of the forced harmonic oscillator is

$$S(x) = \int_{\boldsymbol{T}} dt \ \left(\frac{m}{2} \dot{x}^2(t) - \frac{m}{2} \omega^2 x^2(t) - \lambda f(t) x(t) \right) . \tag{3.68}$$

The forcing term $f(t)$ is assumed to be without physical dimension. Therefore the physical dimension of $\lambda/\hbar$ is $L^{-1}T^{-1}$ when $x(t)$ is of dimension L.

We choose the quadratic form $Q(x)$ to be

$$Q(x) := Q_0(x) + Q_1(x) = \frac{m}{h} \int_{\boldsymbol{T}} dt \ (\dot{x}^2(t) - \omega^2 x^2(t)), \quad h = 2\pi\hbar \tag{3.69}$$

and the potential contribution

$$\int_{\boldsymbol{T}} dt \ V(x(t)) = 2\pi \frac{\lambda}{h} \langle f, x \rangle . \tag{3.70}$$

The quantum mechanical transition amplitudes are given by path integrals of the type (3.66). To begin with we compute

$$I := \int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x) \ \exp\left(-2\pi\imath \frac{\lambda}{h} \langle f, x \rangle \right) \tag{3.71}$$

where

$$d\gamma_{0,0}(x) \overset{\int}{=} \mathcal{D}_Q(x) \ \exp(\pi\imath Q(x)) \tag{3.72}$$

is defined by

$$\int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x)\ \exp(-2\pi\imath\,\langle x', x\rangle) = \gamma_{0,0}(\boldsymbol{X}_{0,0})\ \exp(-\imath\pi W_{0,0}(x')). \quad (3.73)$$

The kernel $G_{0,0}$ of $W_{0,0}$ is the unique Green's function of the differential operator D defined by $Q(x)$ on $\boldsymbol{X}_{0,0}$ (See Appendix IE)

$$Q(x) = \langle Dx, x\rangle$$

$$D = -\frac{m}{h}\left(\frac{d^2}{dt^2} + \omega^2\right) \quad (3.74)$$

$$\begin{aligned} G_{0,0}(r,s) \quad = \quad & \frac{h}{m}\frac{1}{\omega}\theta(s-r)\sin\omega(r-t_a)\ \frac{1}{\sin\omega(t_a-t_b)}\ \sin\omega(t_b-s) \\ & -\frac{h}{m}\frac{1}{\omega}\theta(r-s)\sin\omega(r-t_b)\ \frac{1}{\sin\omega(t_b-t_a)}\ \sin\omega(t_a-s), \end{aligned} \quad (3.75)$$

where θ is the Heaviside step function equal to 1 for positive arguments and zero otherwise.

The gaussian volume of $\boldsymbol{X}_{0,0} \subset \boldsymbol{X}_{b=0}$ is given by (3.39) in terms of the kernel of W_b defining the gaussian γ on $\boldsymbol{X}_{b=0}$ by (3.35), i.e. in terms of the Green's function (see Appendix IE (4.21a))

$$\begin{aligned} G_b(r,s) \quad = \quad & \frac{h}{m}\frac{1}{\omega}\theta(s-r)\cos\omega(r-t_a)\ \frac{1}{\cos\omega(t_a-t_b)}\ \sin\omega(t_b-s) \\ & -\frac{h}{m}\frac{1}{\omega}\theta(r-s)\sin\omega(r-t_b)\ \frac{1}{\cos\omega(t_b-t_a)}\ \cos\omega(t_a-s), \end{aligned} \quad (3.76)$$

$$G_b(t_a, t_a) = \frac{h}{m}\frac{1}{\omega}\sin\omega(t_b-t_a)\frac{1}{\cos\omega(t_b-t_a)} \quad (3.77)$$

$$\gamma_{0,0}(\boldsymbol{X}_{0,0}) = \left(\frac{m\omega}{\imath h}\frac{\cos\omega(t_b-t_a)}{\sin\omega(t_b-t_a)}\right)^{1/2} \quad (3.78)$$

• *A quick calculation of (3.47)*

The integrand being the exponential of a linear functional we can use the Fourier transform (3.49) of the volume element with $x' = \frac{\lambda}{h}f$

$$I = \gamma_{0,0}(\boldsymbol{X}_{0,0})\ \exp\left(-\imath\pi W_{0,0}\left(\frac{\lambda}{h}f\right)\right) \quad (3.79)$$

with

$$W_{0,0}\left(\frac{\lambda}{h}f\right) = \left(\frac{\lambda}{h}\right)^2 \int_{\boldsymbol{T}} dr \int_{\boldsymbol{T}} ds\ f(r)\ f(s)\ G_{0,0}(r,s) \tag{3.80}$$

given explicitly by (3.78) and (3.75). Finally bringing together these equations with (3.66) (3.67), one obtains

$$\begin{aligned}\langle 0, t_b | 0, t_a\rangle &= \left(\frac{m\omega}{\imath h \sin\omega(t_b - t_a)}\right)^{1/2} \\ &\quad \times \exp\left(-\imath\pi \left(\frac{\lambda}{h}\right)^2 \int_{\boldsymbol{T}} dr \int_{\boldsymbol{T}} ds\ f(r) f(s) G_{00}(r,s)\right).\end{aligned} \tag{3.81}$$

This amplitude is identical with the amplitude computed by L. S. Schulman [4] when $x(t_a) = 0$, $x(t_b) = 0$.

• *A general technique*

The quick calculation of (3.47) does not display the power of linear maps. We now compute the more general expression

$$I := \int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x)\ \exp\left(-2\pi\imath\frac{\lambda}{h}\int_{\boldsymbol{T}} dt\ V(x(t))\right) \tag{3.82}$$

following the traditional method

i) expand the exponential

$$\begin{aligned} I &=: \sum I_n \\ I_n &= \frac{1}{n!}\left(\frac{\lambda}{\imath\hbar}\right)^n \int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x) \left(\int_{\boldsymbol{T}} dt\ V(x(t))\right)^n \end{aligned} \tag{3.83}$$

ii) exchange the order of integrations

$$I_n = \frac{1}{n!}\left(\frac{\lambda}{\imath\hbar}\right)^n \int_{\boldsymbol{T}} dt_1 \ldots \int_{\boldsymbol{T}} dt_n \int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x) V(x(t_1)) \ldots V(x(t_n)) \tag{3.84}$$

If $V(x(t))$ is a polynomial in $x(t)$, use a straightforward generalization of the polarization formula (2.47**??**) for computing gaussian integrals of multilinear polynomials.

If $V(x(t)) = f(t)x(t)$, then

$$I_n = \frac{1}{n!}\left(\frac{\lambda}{\imath\hbar}\right)^n \int_{\boldsymbol{T}} dt_1 f(t_1)\dots\int_{\boldsymbol{T}} dt_n f(t_n) \\ \times \int_{\mathbb{X}_{0,0}} d\gamma_{0,0}(x)\, \langle \delta_{t_1}, x\rangle \dots \langle \delta_{t_n}, x\rangle\,. \tag{3.85}$$

Each individual integral I_n can be represented by diagrams as shown after (2.47**??**). A line $G(r,s)$ is now attached to $f(r)$ and $f(s)$ which encode the potential. Each term $f(r)$ is attached to a vertex.

The "quick calculation" bypassed the expansion of the exponential and the (tricky) combinatorics of the polarization formula.

Remarks

- It has been proved[5] that the gaussian covariance is the Feynman propagator for the action functional.
- On the space $\boldsymbol{X}_{0,0}$, the paths are loops, and the expansion $I = \sum_n I_n$ is often called *loop expansion*. An expansion terminating at I_n is said to be n–loop order. The physical dimension analysis of the integral over $\boldsymbol{X}_{0,0}$ shows that the loop expansion is an expansion in powers of h. Indeed a line representing G is of order h, a vertex is of order h^{-1} [See (3.75, 3.70)]. Let L be the number of lines, V the number of vertices, and K the number of independent closed loops, then

  $$L - V = K - 1.$$

  Therefore every diagram with K independent closed loops has a value proportional to h^{k-1}.

3.4 Phase space path integrals

The example in section 3.3 begins with the lagrangian (3.44) of the system; the paths take their values in a configuration space. In this section we construct gaussian path integrals over paths taking their values in phase space. As before, the domain of integration is not the limit $n = \infty$ of $\mathbb{R}^{2n}$, but a function space. The method of discretizing path integrals presented in section 3.1 provides a comparison for earlier heuristic results obtained by replacing a path with a finite number of its values

[6, 7, 8].

Notation

$(\mathbb{M}^D, g)$: the configuration space of the system, metric g,
$T\mathbb{M}^D$: the tangent bundle over $\mathbb{M}$
$T^*\mathbb{M}^D$: the cotangent bundle over $\mathbb{M}$, i.e. the phase space
$L : T\mathbb{M} \longrightarrow \mathbb{R}$: the lagrangian
$H : T^*\mathbb{M} \longrightarrow \mathbb{R}$: the hamiltonian
(q, p): a classical path in phase space
(x, y): an arbitrary path in phase space,

$$x : \mathbb{T} \longrightarrow \mathbb{R}^D \ , \quad y : \mathbb{T} \longrightarrow \mathbb{R}_D \ , \quad \mathbb{T} = [t_a, t_b]$$

A path (x, y) is characterized by D initial vanishing boundary conditions, and D final vanishing boundary conditions.
$\theta := p_i \, dq^i - H \, dt$, the canonical 1-form (a relative integral invariant of the hamiltonian Pfaff system)
$F := d\theta$, the canonical 2-form, a symplectic form on $\mathbb{T}^*\mathbb{M}$ (see section 6 for phase space in the language of symplectic geometry.)

The phase space action functional is

$$S(x, y) := \int_{\mathbb{T}} y(t) \ dx(t) - H(y(t), x(t), t) \ dt \tag{3.86}$$

The action functions, solutions of the Hamilton-Jacobi equation for the various boundary conditions,

$$x(t_a) = x_a \ , \quad x(t_b) = x_b \ , \quad p(t_a) = p_a \ , \quad p(t_b) = p_b,$$

are

$$\begin{aligned}
\mathcal{S}(x_b, x_a) &= S(q, p) && (3.87)\\
\mathcal{S}(x_b, p_a) &= S(q, p) + \langle p_a, q(t_a) \rangle && (3.88)\\
\mathcal{S}(p_b, x_a) &= S(q, p) - \langle p_b, q(t_b) \rangle && (3.89)\\
\mathcal{S}(p_a, p_b) &= S(q, p) - \langle p_b, q(t_b) \rangle + \langle p_a, q(t_a) \rangle \, . && (3.90)
\end{aligned}$$

Jacobi operator

The Jacobi operator in configuration space is obtained by varying a 1-parameter family of paths in the action functional (Appendix IE). The Jacobi operator in phase space is obtained by varying a 2-parameter family of paths in the phase space action functional. Let

$$u, v \in [0, 1]$$

and let

$$\bar{\gamma}(u,v) : \mathbb{T} \longrightarrow T^*\mathbb{M}$$
$$\text{by} \left\{ \begin{array}{lll} \bar{\alpha}(u) : \mathbb{T} \to \mathbb{M}, & \bar{\alpha}(0) = q, & \bar{\alpha}(1) = x \\ \bar{\beta}(u,v) : \mathbb{T} \to T^*\mathbb{M}, & \bar{\beta}(0,0) = p, & \bar{\beta}(1,1) = y \end{array} \right. \tag{3.91}$$

$$\bar{\alpha}(u)(t) = \alpha(u,t)\ , \quad \bar{\beta}(u,v)(t) = \beta(u,v,t).$$

For $\mathbb{M}^D = \mathbb{R}^D$, the family $\bar{\beta}$ depends only on v.

Let ζ be the $2D$ dimensional vector (D contravariant, D covariant components):

$$\zeta := \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where

$$\xi := \left. \frac{d\bar{\alpha}(u)}{du} \right|_{u=0} \qquad \eta := \left. \frac{\partial \bar{\beta}(u,v)}{\partial v} \right|_{v=0}.$$

The expansion of the action functional $S(x,y)$ around $S(q,p)$ is

$$(S \circ \bar{\gamma})(1,1) = \sum_{n=0}^{\infty} \frac{1}{n!} (S \circ \bar{\gamma})^{(n)}(0,0).$$

The first variation vanishes for paths satisfying the Hamilton set of equations. The second variation defines the Jacobi operator.

Example: The phase space Jacobi equation in $\mathbb{R}^D \times \mathbb{R}_D$. The expansion of the action functional (3.86) gives the following Jacobi equation

$$\begin{pmatrix} -\partial^2 H/\partial q^\alpha \partial q^\beta & -\partial/\partial t - \partial^2 H/\partial q^\alpha \partial p_\beta \\ \partial/\partial t - \partial^2 H/\partial p_\alpha d q^\beta & -\partial^2 H/\partial p_\alpha \partial p_\beta \end{pmatrix} \begin{pmatrix} \xi^\beta \\ \eta_\beta \end{pmatrix} = 0.$$

Example: A free particle in the riemannian manifold $\mathbb{M}^D$ with metric g. Varying u results in changing the fibre $T^*_{\alpha(u,t)}\mathbb{M}^D$; the momentum $\beta(u,v,t)$ is conveniently chosen to be the momentum parallel transported along a geodesic beginning at $\alpha(0,t)$; the momentum is then uniquely defined by

$$\nabla_u \beta = 0,$$

the vanishing of the covariant derivative of β along the path $\alpha(\cdot,t) : u \mapsto \alpha(u,t)$.

The action functional is

$$S(x,y) = \int_{\boldsymbol{T}} dt \left(\langle p(t), \dot{q}(t) \rangle - \frac{1}{2m} (p(t)|p(t)) \right)$$

The bracket $\langle , \rangle$ is the duality pairing of $T\mathbb{M}$ and $T^*\mathbb{M}$ and the parenthesis $(|)$ is the scalar product defined by the inverse metric g^{-1}.

The Jacobi operator is

$$\mathcal{J}(q,p) = \begin{pmatrix} -\frac{1}{m} R^{\delta}_{\alpha\beta\gamma}\ g^{\beta\epsilon}\ p_\epsilon\ p_\delta & -\delta^\gamma_\alpha\ \nabla_t \\ \delta^\alpha_\gamma\ \nabla_t & -\frac{1}{m} g^{\alpha\gamma} \end{pmatrix}$$

Covariances

The second variation of the action functional in phase space provides a quadratic form Q on the space parameterized by (ξ,η) with

$$\xi \in T_x\mathbb{M}^D \ , \quad \eta \in T^*_y\mathbb{M}^D$$

$$Q(\xi,\eta) = \langle \mathcal{J}(q,p) \cdot (\xi,\eta), (\xi,\eta) \rangle$$

where $\mathcal{J}(q,p)$ is the Jacobi operator defined by a classical path (q,p). There exists a quadratic form W, corresponding to the quadratic form Q; it is defined by the Green functions of the Jacobi operator

$$W(\xi',\eta') = (\xi'_\alpha, \eta'^\alpha) \begin{pmatrix} G_1^{\alpha\beta} & G_2{}^\alpha{}_\beta \\ G^3_\alpha{}^\beta & G^4_{\alpha\beta} \end{pmatrix} \begin{pmatrix} \xi'_\beta \\ \eta'^\beta \end{pmatrix}$$

where

$$\mathcal{J}_r(q,p)G(r,s) = \mathbb{1}\delta_s$$

$\mathcal{J}_r$ acts on the r-argument of the $2D \times 2D$ matrix G made of the 4 blocks (G_1, G_2, G^3, G^4)

Once the variance W and the covariance G are identified, path integrals over phase space are constructed as in the previous examples. For explicit expressions which include normalization, correspondences with configuration space path integrals, discretization of phase space integrals, physical interpretations of the covariances in phase space, and infinite dimensional Louiville volume elements, see, for instance, John LaChapelle's Ph.D. dissertation [9] "Functional Integration on Symplectic Manifolds."

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