2

First Lesson: Gaussian Integrals

Given the experience accumulated since Feynman's doctoral thesis, the time has come to extract a simple and robust axiomatics for functional integration from the work done during the past sixty years, and to investigate approaches other than the ones dictated by an action functional.

Here, "simple and robust" means easy and safe techniques for computing integrals by integration by parts, change of variable of integration, expansions, approximations etc ....

We begin with Gaussian integrals in $\mathbb{R}$ and $\mathbb{R}^D$, and define them by an equation which can be readily extended to Gaussians in Banach spaces $\boldsymbol{X}$.

2.1 Gaussians in $\mathbb{R}$

A Gaussian random variable, and its concomitant the Gaussian volume element, are marvelous multifaceted tools. We summarize their properties in Appendix C; in this section we present only properties leading to volume elements of particular interest in functional integrals.

2.2 Gaussians in $\mathbb{R}^D$

Let

$$I_D(a) := \int_{\mathbb{R}^D} dx \, \exp\Big(-\frac{\pi}{a}\,|x|^2\Big) \quad \text{for} \quad a > 0, \tag{2.1}$$

with $dx := dx^1 \cdots dx^D$ and $|x|^2 = \sum_{j=1}^{D} \left(x^j\right)^2 = \delta_{ij} x^i x^j$. From elementary calculus, one gets

$$I_D(a) = a^{D/2}. \tag{2.2}$$

Therefore when $D = \infty$,

$$I_\infty(a) = \begin{cases} 0 & \text{if} \quad 0 < a < 1 \\ 1 & \qquad a = 1 \\ \infty & \qquad 1 < a \end{cases} . \tag{2.3}$$

Clearly an unsatisfactory situation; it can be corrected by introducing a volume element $D_a x$ scaled by the parameter a as follows:

$$D_a x := \frac{1}{a^{D/2}} dx^1 \cdots dx^D. \tag{2.4}$$

The volume element $D_a x$ can be characterized by the integral

$$\int_{\mathbb{R}^D} D_a x \, \exp\Bigl(-\frac{\pi}{a} |x|^2 - 2\pi\imath \langle x', x \rangle\Bigr) := \exp\Bigl(-a\pi |x'|^2\Bigr), \tag{2.5}$$

where x' is in the dual $\mathbb{R}_D$ of $\mathbb{R}^D$. A point $x \in \mathbb{R}^D$ is a contravariant vector. A point $x' \in \mathbb{R}_D$ is a covariant vector.

The integral (2.5) suggest the definition of a volume element $d\gamma_a(x)$ by the following equation:

$$\int_{\mathbb{R}^D} d\gamma_a(x) \, \exp(-2\pi\imath \langle x', x \rangle) := \exp\Bigl(-a\pi |x'|^2\Bigr). \tag{2.6}$$

Here we can write

$$d\gamma_a(x) = D_a x \, \exp\Bigl(-\frac{\pi}{a} |x|^2\Bigr), \tag{2.7}$$

but this equality is meaningless in infinite dimensions. However, the integral (2.5) and (2.6) remain meaningful. We introduce a different equality symbol, namely

$$\overset{\int}{=}, \tag{2.8}$$

a qualified equality in integration theory; e.g.

$$d\gamma_a(x) \overset{\int}{=} D_a x \; \exp\left(-\frac{\pi}{a} |x|^2\right). \tag{2.9}$$

are defined by the same integral.

A linear map $A : \mathbb{R}^D \to \mathbb{R}^D$ by

$$y = Ax, \quad \text{i.e.} \quad y^j = A^j{}_i \, x^i, \tag{2.10}$$

generalizes the quadratic form $\delta_{ij} y^i y^j$ to a general positive quadratic form

$$Q(x) = \delta_{ij} \, A^i{}_k \, A^j{}_\ell \, x^k \, x^\ell =: Q_{k\ell} \, x^k \, x^\ell. \tag{2.11}$$

Therefore a linear change of variable in the integral (2.5) can be used for defining Gaussian volume element $d\gamma_Q$ with respect to the quadratic for (2.11). We begin with the definition

$$\int_{\mathbb{R}^D} D_a y \exp\Big(-\frac{\pi}{a} |y|^2 - 2\pi\imath \langle y', y\rangle\Big) := \exp\Big(-a\pi |y'|^2\Big). \tag{2.12}$$

Under the change of variable $y = ax$,

$$D_a y = a^{-D/2} dy^1 \wedge \cdots \wedge dy^D$$

becomes

$$\begin{aligned} D_{a,Q} x &= a^{-D/2} \det A dx^1 \wedge \cdots \wedge dx^D \\ &= |\det Q/a|^{1/2} \, dx^1 \cdots dx^D. \end{aligned} \tag{2.13}$$

The change of variable

$$y'_j = x'_i B^i{}_j, \tag{2.14}$$

defined by

$$\langle y', y\rangle = \langle x', x\rangle, \quad \text{i.e.} \quad y'_j y^j = x'_i x^i \tag{2.15}$$

implies

$$B^i{}_j A^j{}_k = \delta^i{}_k. \tag{2.16}$$

Equation (2.12) now reads

$$\int_{\mathbb{R}^D} D_{a,Q} x \, \exp\Big(-\frac{\pi}{a} Q(x) - 2\pi\imath \langle x', x\rangle\Big) := \exp(-a\pi W(x')), \tag{2.17}$$

where

$$\begin{aligned} W(x') &= \delta^{ij} x'_k x'_\ell B^k{}_i B^\ell{}_j, \\ &=: x'_k x'_\ell W^{k\ell}. \end{aligned} \tag{2.18}$$

$W^{k\ell}$ on $\mathbb{R}_D$ can be said to be "the inverse" of $Q_{k\ell}$ on $\mathbb{R}^D$ because the matrices A and B are inverse of each other.

In conclusion, in $I\!R^D$, the gaussian volume element defined in (2.17) by the quadratic form aW is

$$d_{\gamma_{a,Q}}(x) = \mathcal{D}_{a,Q}x \cdot \exp\Big(-\frac{\pi}{a}Q(x)\Big) \tag{2.19}$$

$$= dx^1 \dots dx^D \left(\det_{k,\ell} \frac{Q_{k\ell}}{a}\right)^{1/2} \exp\Big(-\frac{\pi}{a}Q(x)\Big)\,. \tag{2.20}$$

Remark: The volume element $\mathcal{D}_a x$ has been chosen so as to be without physical dimension. In Feynman's dissertation, the volume element $\mathcal{D}x$ is the limit for $D = \infty$ of the dicretized expression

$$\mathcal{D}x = \prod_i dx(t_i)\, A^{-1}(\delta t_i)\,. \tag{2.21}$$

The normalization factor was determined by requiring that the wave function for a free particle of mass m moving in one dimension be continuous. It was found to be

$$A(\delta t_k) = (2\pi\imath\hbar\delta t_k/m)^{1/2}\,. \tag{2.22}$$

A general expression for the absolute value of the normalization factor was determined by requiring that the short time propagators be unitary. For a system with action function $\mathcal{S}$, and paths taking their values in an n-dimensional configuration space,

$$|A(\delta t_k)| = \left|\det_{\mu,\nu} \partial^2\mathcal{S}(x^\mu(t_{k+1})\,, x^\nu(t_k))\,/\partial x^\mu(t_{k+1})\,\partial x^\nu(t_k)\right|^{1/2}\,. \tag{2.23}$$

The "intractable" product of the infinite number of normalization factors was found to be a Jacobian later encountered by integrating out momenta from phase space path integrals. Equation (2.5) suggests equation (2.17) and equation (2.22) in which $\mathcal{D}_{a,Q}(x)$ provides a volume element obtained directly without working through an infinite product of short time propagators.

2.3 Gaussian in Banach Spaces

In infinite dimensions one is often confronted with situations in which Q does not have a unique inverse, or with situations in which Q is degenerate. Therefore we shall make sense of the definition (2.17) without introducing (2.14) the map $B : \boldsymbol{X}' \to \boldsymbol{Y}'$, inverse of $A : \boldsymbol{X} \to \boldsymbol{Y}$. The strategy is to exploit the properties of dual spaces. Let $\boldsymbol{X}$ be the domain of integration of a functional integral; provided $\boldsymbol{X}$ is a linear space, one can define its dual $\boldsymbol{X}'$, namely the space of linear maps on $\boldsymbol{X}$. Let

$$A : \boldsymbol{X} \to \boldsymbol{Y} \quad \text{by} \quad Ax = y,$$

linearly, and let $\tilde{A}$ be the transpose of A defined by

$$\tilde{A} : \boldsymbol{Y}' \to \boldsymbol{X}' \quad \text{by} \quad \tilde{A}y' = x'$$

such that

$$\left\langle \tilde{A}y', x \right\rangle = \langle y', Ax \rangle .$$

The matrix $B : \boldsymbol{X}' \to \boldsymbol{Y}'$ is replaced by $\tilde{A} : \boldsymbol{Y}' \to \boldsymbol{X}'$.

The defining equation (2.17) defines also the Gaussian volume element

$$d\gamma_{a,Q}x \overset{\int}{=} D_{a,Q}x \exp\Big(-\frac{\pi}{a}Q(x)\Big) \tag{2.24}$$

by its Fourier transform $\mathcal{F}\gamma_{a,Q}$, i.e. by the quadratic form W on $\mathbb{R}_D$. Equation (2.17) has a straightforward generalization to Gaussian on a Banach space $\boldsymbol{X}$.

Definition

A Gaussian volume element $d\gamma_{a,Q}$ on a vector space $\boldsymbol{X}$ can be defined by its Fourier transform:

$$\int_{\mathbb{X}} d\gamma_{a,Q}(x) \exp(-2\pi\imath \langle x', x \rangle) = \exp(-a\pi W(x')) \tag{2.25}$$

where† $\boldsymbol{X}$ is a vector space, $\boldsymbol{X}'$ its dual, and $x' \in \boldsymbol{X}'$; W is a quadratic form on $\boldsymbol{X}'$, and Q a quadratic form on $\boldsymbol{X}$. Set

$$W(x') =: \langle x', Gx' \rangle , \quad \text{and} \quad Q(x) =: \langle Dx, x \rangle ; \tag{2.26}$$

W and Q are said to be inverse of each other when D and G are inverse of each other

$$DG = 1, \quad GD = 1. \tag{2.27}$$

† The label Q is introduced for later use. The quadratic form Q is not needed for establishing the moment and the polarization formulae nor the diagram expansion of a Gaussian integral.

Previously we worked with $d\gamma_{a,Q}$ where a is a positive number (2.24). *As long as the Gaussian is defined by its Fourier transform we can replace a by $s \in \{1, i\}$.*

So far, the Banach space $\boldsymbol{X}$ is thought of as a space of paths

$$x : \mathbb{T} \longrightarrow \mathbb{M}^D \tag{2.28}$$

But equations (2.25) and (2.27) remain valid when the variable of integration is a field ψ on $\mathbb{R}^D$ with a euclidean or a minkowski signature.

$$\psi : \mathbb{R}^D \longrightarrow \mathbb{M}^D \tag{2.29}$$

The volume element definition corresponding to (2.24) (2.25) can be written

$$\int \mathcal{D}\psi \, \exp\Big(\frac{\imath}{\hbar} S(\psi) - \imath \langle J, \psi \rangle\Big) = \exp\Big(\frac{\imath}{\hbar} W(J)\Big) = Z(J)\,, \tag{2.30}$$

where ψ is either a self-interacting field, or a collection of interacting fields. But the generating functional $Z(J)$ is difficult to ascertain *a priori* for the following reason. Let $\Gamma(\bar{\psi})$ be the Legendre transform of $W(J)$:

$$\hbar\bar{\psi} := \frac{\delta W(J)}{\delta J}, \qquad \Gamma(\bar{\psi}) := W\big(J(\bar{\psi})\big) - \hbar \big\langle J(\bar{\psi})\,, \bar{\psi} \big\rangle\,. \tag{2.31}$$

Then $\Gamma(\bar{\psi})$ is the inverse of $W!\,(J)$ in the same sense as Q and W are inverse of each other (2.24), but $\Gamma(\bar{\psi})$ is the *effective action* which has to be used for computing observables. If $S(\psi)$ is quadratic, the *bare action* $S(\psi)$ and the effective action $\Gamma(\psi)$ are identical, the fields do not interact. But in the case of interacting fields, the exact relation between bare and effective action is the main problem of quantum field theory (see Chapters **??**14, **??**15, **??**16).

In this chapter we define a volume element on the space Φ of fields ϕ by the equation

$$\int_\Phi \mathcal{D}_Q\phi \cdot \exp\Big(-\frac{\pi}{s} Q(\phi)\Big) \, \exp(-2\pi\imath \langle J, \phi \rangle) := \exp[-\pi s W(J)]\,, \tag{2.32}$$

or more conveniently $d\mu_G$ by

$$\int_\Phi d\mu_G(\phi)\, \exp(-2\pi\imath \langle J, \phi\rangle) := \exp[-\pi s W(J)] \tag{2.33}$$

for a given Q and W inverse of each other, and exploit this definition. As before the covariance G is defined by

$$W(J) = \langle J, GJ\rangle\,; \tag{2.34}$$

it is the inverse of the operator D defined by

$$Q(\phi) = \langle D\phi, \phi\rangle\,; \tag{2.35}$$

it is also the two-point function

$$\frac{s}{2\pi} G(x,y) = \int_\Phi d\mu_G(\phi)\, \phi(x)\, \phi(y)\,. \tag{2.36}$$

We shall construct covariances in quantum mechanics and quantum field theory on two simple examples.
In Quantum Mechanics:

Let $D = -\frac{d^2}{dt^2}$; its inverse on the space X_{ab} of paths with two fixed end points is

$$\begin{aligned} G(t,s) \;\; &= \;\; \theta\,(s-t)\,(t-t_a)\,(t_a-t_b)^{-1}\,(t_b-s) \\ &\quad -\theta\,(t-s)\,(t-t_b)\,(t_b-t_a)^{-1}\,(t_a-s)\,. \end{aligned} \tag{2.37}$$

In Quantum Field Theory:

Let $D = -\Delta$ on $I\!R^D$; then

$$G(x,y) = \frac{C_D}{|x-y|^{D-2}}, \tag{2.38}$$

with a constant C_D equal to

$$\Gamma\left(\frac{D}{2}-1\right)/4\pi^{D/2}. \tag{2.39}$$

Notice that $G(t,s)$ is a continuous function. $G(x,y)$ is singular at the origin for euclidean fields, and singular on the lightcone for minkowskian fields. However, we note that the quantity of interest is not the covariance G, but the variance W:

$$W(J) = \langle J, GJ\rangle\,, \tag{2.40}$$

which is singular only if J is a point-like source $\langle J, \phi\rangle = \text{constant}\cdot\phi(x)$.

2.4 Variances and covariances

The quadratic form W on $\boldsymbol{X}'$ that characterizes the Fourier transform $\mathcal{F}\gamma_{s,Q}$ of the Gaussian which in turn characterizes the Gaussian $\gamma_{s,Q}$ is known in probability theory as the variance, and (2.26) the kernel G as the covariance of the Gaussian distribution. In quantum theory G is the propagator of the system. It is also the "two-point function" since (2.45) gives

$$\int_{\mathbb{X}} d\gamma_{s,Q} \langle x'_1, x\rangle \langle x'_2, x\rangle = \frac{s}{2\pi} W(x'_1, x'_2)\,. \tag{2.41}$$

The exponential $\exp(-s\pi W(x'))$ is a generating functional which yields the moments (2.43) and (2.44) and the polarization (2.47). It has been extensively used by Schwinger who considers the term $\langle x', x\rangle$ as a source.

In this section, we work only with the variance W. In section **??**3.4 we work Gaussian volume elements, i.e. with the quadratic form Q on $\boldsymbol{X}$; i.e. we move from the algebraic theory of Gaussians (section 2.3) to their differential theory, (section **??**3) which is commonly used in physics.

Moments

The integral of polynomials with respect to a Gaussian volume element follows readily from the definition (2.25) after replacing x' by $\frac{c}{2\pi\imath}x'$, i.e.

$$\int_{\mathbb{X}} d\gamma_{s,Q}(x) \exp(-c\langle x', x\rangle) = \exp\bigl(c^2 s W(x')\,/4\pi\bigr)\,. \tag{2.42}$$

Expanding both sides in powers of c, yields

$$\int_{\mathbb{X}} d\gamma_{s,Q}(x) \langle x', x\rangle^{2n+1} = 0 \tag{2.43}$$

and

$$\begin{aligned}\int_{\mathbb{X}} d\gamma_{s,Q}(x) \langle x', x\rangle^{2n} &= \frac{2n!}{n!}\left(\frac{sW(x')}{4\pi}\right)^n \\ &= \frac{2n!}{2^n n!}\left(\frac{s}{2\pi}\right)^n W(x')^n \end{aligned}\tag{2.44}$$

Hint: $W(x')$ is an abbreviation of $W(x', x')$, therefore n-th order terms in expanding the r.h.s. are equal to $2n$-th order terms of the l.h.s.

Polarization.1

The integral of a multilinear expression,

$$\int_{\mathbb{X}} d\gamma_{s,Q}(x)\, \langle x'_1, x\rangle \cdots \langle x'_{2n}, x\rangle \tag{2.45}$$

can be readily be computed; replacing x' in the definition (2.25) by the linear combination $c_1 x'_1 + \cdots + c_{2n} x'_{2n}$ and equating the $(c_1, c_2, \cdots, c_{2n})$-terms in both sides of the equation yields

$$\begin{aligned} &\int_{\mathbb{X}} d\gamma_{s,Q}(x)\, \langle x'_1, x\rangle \cdots \langle x'_{2n}, x\rangle \\ &= \frac{1}{2^n n!} \left(\frac{s}{2\pi}\right)^n \sum W\left(x'_{i_1}, x'_{i_2}\right) \cdots W\left(x'_{i_{2n-1}}, x'_{i_{2n}}\right), \end{aligned} \tag{2.46}$$

where the sum is performed over all possible distributions of the arguments. However there are $2^n n!$ identical terms in this sum since $W\left(x'_{i_j}, x'_{i_k}\right) = W\left(x'_{i_k}, x'_{i_j}\right)$ and since the product order is irrelevant. Finally

$$\begin{aligned} &\int_{\mathbb{X}} d\gamma_{s,Q}(x)\, \langle x'_1, x\rangle \cdots \langle x'_{2n}, x\rangle \\ &= \left(\frac{s}{2\pi}\right)^n \sum W\left(x'_{i_1}, x'_{i_2}\right) \cdots W\left(x'_{i_{2n-1}}, x'_{i_{2n}}\right), \end{aligned} \tag{2.47}$$

without repetition of identical terms in the sum.

Example If $2n = 4$, the sum consists of three terms which can be recorded by three diagrams as follows. Let $1, 2, 3, 4$ designate x'_1, x'_2, x'_3, x'_4 respectively, and a line from i_1 to i_2 records $W\left(x'_{i_1}, x'_{i_2}\right)$ then the sum in (2.47) is recorded by the three diagrams.

Fig. 2.1. Diagrams

Polarization.2

We anticipate on Chapter **??** and give a proof of the polarization formula (2.47) in terms of Q. i.e. we use the qualified equality

$$d\gamma_{s,Q}(x) \stackrel{\int}{=} \mathcal{D}_{s,Q}(x)\, \exp\Big(-\frac{\pi}{s}Q(x)\Big)\,, \tag{2.48}$$

where the quadratic form Q on $\boldsymbol{X}$ is defined as follows. Let

$$W(x') =: \langle x', Gx'\rangle \quad x' \in \boldsymbol{X}. \tag{2.49}$$

Let D be the differential operator on $\boldsymbol{X}$ such that

$$DG = \mathbb{1}_{\mathbb{X}}, \quad \text{and} \quad GD = \mathbb{1}_{\mathbb{X}} \tag{2.50}$$

and define

$$Q(x) := \langle Dx, x\rangle\,. \tag{2.51}$$

The basic integration by parts formula

$$\begin{aligned} &\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x)\, \exp\Big(-\frac{2\pi}{s}S(x)\Big)\, \frac{\delta F(x)}{\delta x(t)} \\ := \quad &-\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x)\, \exp\Big(-\frac{2\pi}{s}S(x)\Big)\, F(x)\, \frac{\delta}{\delta x(t)}\Big(-\frac{2\pi}{s}S(x)\Big) \end{aligned} \tag{2.52}$$

yields the polarization formula (2.47) when

$$S(x) = \frac{1}{2}Q(x) = \int dr\, Dx(r)\cdot x\,(r)\,. \tag{2.53}$$

Indeed,

$$-\frac{\delta}{\delta x(t)}\frac{\pi}{s}Q(x) = -2\frac{\pi}{s}\int dr\, Dx(r)\,\delta(r-t) = -\frac{2\pi}{s}D_t x(t)\,. \tag{2.54}$$

When

$$F(x) = x(t_1)\ldots x(t_n) \tag{2.55}$$

then

$$\frac{\delta F(x)}{\delta x(t)} = \sum_{i=1}^{n}\delta(t-t_i)\, x(t_1)\ldots \hat{x}(t_i)\ldots x!\,(t_n)\,. \tag{2.56}$$

The n-point function with respect to the quadratic action $S = \frac{1}{2}Q$ is by definition

$$G_n(t_1,\ldots,t_n) := \int \mathcal{D}_{s,Q}(x)\, \exp\Big(-\frac{\pi}{s}Q(x)\Big)\, x(t_1)\ldots x(t_n) \tag{2.57}$$

Therefore the l.h.s. of the integration by parts formula (2.52) is

$$\int_{\mathbb{X}} \mathcal{D}_{s,Q}(x) \, \exp\left(-\frac{\pi}{s} Q(x)\right) \frac{\delta F(x)}{x(t)}$$
$$= \sum_{i=1}^{n} \delta(t-t_i) \, G_{n-1}\bigl(t_1, \ldots, \hat{t}_I, \ldots, t_n\bigr) \, . \tag{2.58}$$

Given (2.54) the r.h.s. of (2.52) is

$$- \int_{\mathbb{X}} \mathcal{D}_{s,Q}(x) \, \exp\Bigl(-\frac{\pi}{s} Q(x)\Bigr) \, x(t_1) \ldots x(t_n) \left(-\frac{2\pi}{s} D_t x(t)\right)$$
$$= \frac{2\pi}{s} D_t G_{n+1}(t, t_1, \ldots, t_n) \, . \tag{2.59}$$

The integration by parts formula (2.52) yields a recurrence formula for the n-point functions, G_n, namely

$$\frac{2\pi}{s} D_t G_{n+1}(t, t_1, \ldots, t_n) = \sum_{i=1}^{n} \delta(t-t_i) \, G_{n-1}\bigl(t_1, \ldots, \hat{t}_I, \ldots, t_n\bigr) \tag{2.60}$$

equivalently (replace n by $n-1$)

$$\frac{2\pi}{s} D_{t_1} G_n(t_1, \ldots, t_n) = \sum_{i=2}^{n} \delta(t_1-t_i) \, G_{n-2}\bigl(\hat{t}_1, \ldots, \hat{t}_i, \ldots, t_n\bigr) \, . \tag{2.61}$$

A similar recurrence formula exists for quantum field theory functional integral; it is exploited in Chapter **??**14 and generalized to actions which are not quadratic.

Linear Maps

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two Banach spaces, possibly two copies of the same space. Let L be a linear continuous map $L : \boldsymbol{X} \to \boldsymbol{Y}$ by $x \to y$ and $\tilde{L} : \boldsymbol{X'} \to \boldsymbol{Y'}$ by $y' \to x'$ defined by

$$\left\langle \tilde{L} y', x \right\rangle = \langle y', Lx \rangle \, . \tag{2.62}$$

Then the Fourier transforms $\mathcal{F}\gamma_{\mathbb{X}}$, $\mathcal{F}\gamma_{\boldsymbol{Y}}$ of Gaussians on $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively satisfy the equation

$$\mathcal{F}\gamma_{\boldsymbol{Y}} = \mathcal{F}\gamma_{\mathbb{X}} \circ \tilde{L}, \tag{2.63}$$

i.e.

$$W_{\boldsymbol{Y'}} = W_{\mathbb{X}'} \circ \tilde{L}. \tag{2.64}$$

The following diagram will be used extensively

Fig. 2.2. linear maps

2.5 Scaling and coarse graining

In this section, we exploit the scaling properties of Gaussian volume elements on spaces $\mathbf{\Phi}$ of fields ϕ on $\mathbb{M}^D$. These properties are valid whether $\mathbb{M}^D$ is a vector space with euclidean or minkowskian signature. These properties are applied to the $\lambda - \phi^4$ system in section **??**16.2.

The Gaussian volume element μ_G is defined by (2.33) or (2.34). The covariance G is the two-point function (2.36). Objects defined by the covariance G include

- the functional Laplacian

$$\Delta_G := \frac{s}{2\pi} \int_{R^D} dx \int_{R^D} dy\, G(x,y)\, \frac{\delta^2}{\delta\phi(x)\,\delta\phi(y)}, \quad s \in \{1, i\}, \tag{2.65}$$

- convolution with volume element μ_G

$$(\mu_G * F)(\phi) := \int_{\mathbb{X}} d\mu_G(\psi)\, F(\phi + \psi)\,, \tag{2.66}$$

- hence

$$\mu_G * F = \exp\left(\frac{1}{2}\Delta_G\right) F, \tag{2.67}$$

- the Bargmann-Segal transform

$$B_G := \mu_G * = \exp\left(\frac{1}{2}\Delta_G\right), \tag{2.68}$$

- and the Wick transform

$$: \quad :_G \quad := \exp\left(-\frac{1}{2}\Delta_G\right). \tag{2.69}$$

Scaling

The scaling properties of covariances can be used for investigating the transformation, or the invariance, of some quantum systems under a change of scale. The strategy consists of the following:

- Introduce an additional, independent scaling variable l. There are two options for the domain of the scaling variable: $l \in [0, \infty[$ or Brydges's choice $l \in [1, \infty[$ together with $l^{-1} \in [1, 0[$.
- Use the new variable l for constructing a parabolic scaling evolution equation in l satisfied by some quantum systems. Techniques developed for the time evolution of diffusion and Schrödinger systems can be applied to scaling evolution equations.

The addition of an independent scaling variable does not affect the symmetries of the system. For example, limiting the range of the independent scaling variable does not affect the symmetries of the domain of a field, be it euclidean or minkowskian.

The scaling operator S_l acting on a function u of physical length dimension $[u]$ is by definition

$$S_l\, u(x) := l^{[u]}\, u\left(\frac{x}{l}\right), \qquad x \in \mathbb{R}. \tag{2.70}$$

A physical dimension is often given in powers of mass, length, and time. Here we set $\hbar = 1$, $c = 1$, and the physical dimensions are physical *length* dimensions. We choose length dimension rather than the more conventional mass dimension because we define fields on coordinate space, not on momentum space. The subscript of the scaling operator has no dimension.

The scaling of an interval $[a, b[$ is given by

$$S_l\, [a, b[\; = \left\{ \frac{s}{l} \middle| s \in [a, b[\right\}, \quad \text{i.e.} \quad S_l\, [a, b[\; = \left[\frac{a}{l}, \frac{b}{l}\right[. \tag{2.71}$$

By definition the (dimensional) scaling of a functional F is

$$(S_l F)(\phi) = F(S_l \phi). \tag{2.72}$$

We use multiplicative differentials which are scale invariant:

$$d^\times l = dl/l \tag{2.73}$$

$$\partial^\times/dl = l\,\partial/\partial l \tag{2.74}$$

A covariance in minkowskian or euclidean space can be written

$$G(|x-y|) = \int_0^\infty d^\times l\, S_l\, u(|x-y|)\,, \tag{2.75}$$

where $[u] = [G] = 2 - D$. If $G(|x-y|)$ is given by,

$$G(|x-y|) = C_D/|x-y|^{D-2}$$

then the only requirement on u in the euclidean case is

$$\int_0^\infty d^\times k \cdot k^{-[u]} u(k^2) = C_D \tag{2.76}$$

and in the minkowski case $u(-k^2) = i^{D-2}u(k^2)$. When a covariance is the Green's function of an operator, the normalization C_D is determined. For example, if $\Delta G = \mathbb{1}$ and $\Delta = \sum \partial^2/\left(\partial x^i\right)^2$,

$$C_D = \frac{\Gamma(D/2)}{(D-2)\,(2\pi)\,D/2}. \tag{2.77}$$

The domain of integration $[0,\infty[$ of the scaling variable can be broken up into a union of subdomains,

$$[0,\infty[= \bigcup_{-\infty}^{\infty} [2^j l_0, 2^{j+1} l_0[, \tag{2.78}$$

which expresses the possibility of separating different scale contributions. The corresponding decompositions of covariance is

$$G = \sum_{j=-\infty}^{+\infty} G_{[2^j l_0, 2^{j+1} l_0[}, \tag{2.79}$$

Here $G(x,y)$ is a function of $|x-y|$, and we write simply $G(|x-y|) =: G(\xi)$; eq. (2.79) decomposes G into self similar covariances in the following sense

$$G_{[a,b[}(\xi) := \int_{[a,b[} d^\times s \cdot S_s u(\xi) \tag{2.80}$$

$$= \int_{S_l[a,b[} S_l\left(d^\times s \cdot S_s u\,(\xi)\right) \tag{2.81}$$

$$= \int_{a/l}^{b/l} d^{\times} s\, S_s\, l^{2[\phi]}\, u(\xi/l) \quad \text{using (2.70)}$$
$$= l^{[2\phi]}\, G_{[a/l,b/l[}(\xi/l) \quad \text{using (2.80).} \tag{2.82}$$

Henceforth the suffix G in the objects defined by covariances such as μ_G, Δ_G, B_G, $: \ :_G$ is replaced by the interval defining the scale dependent covariance; for example

$$\mu_{[l_0 \infty[} = \mu_{[l_0,l[} * \mu_{[l,\infty[}. \tag{2.83}$$

It follows from the definition (2.33) of the Gaussian volume element μ_G and the definition (2.34) of the covariance G, that a scale decomposition of the covariance corresponds to a scale decomposition of the fields. Indeed, if $G = G_1 + G_2$ then $W = W_1 + W_2$, $\mu_G = \mu_{G_1} * \mu_{G_2} = \mu_{G_2} * \mu_{G_1}$ and

$$\int_{\mathbf{\Phi}} d\mu_G(\phi) \exp(-2\pi\imath \langle J, \phi \rangle)$$
$$= \int_{\mathbf{\Phi}} d\mu_{G_2}(\phi_2) \int_{\mathbf{\Phi}} d\mu_{G_1}(\phi_1) \exp(-2\pi\imath \langle J, \phi_1 + \phi_2 \rangle) \tag{2.84}$$

and $\phi = \phi_1 + \phi_2$.

To the covariance decomposition (2.79) corresponds to the field decomposition

$$\phi = \sum_{j=-\infty}^{+\infty} \phi_{[2^j l_0, 2^{j+1} l_0[} \tag{2.85}$$

We also write

$$\phi(x) = \sum_{j=-\infty}^{+\infty} \phi_j(l_0, x)\,. \tag{2.86}$$

2.5.1 Brydges coarse-graining operator P_l

D. Brydges, J. Dimock, and T.R. Hurd introduced and developed the properties of a coarse graining operator P_L which rescales the Bargmann-

Segal transform so that all integrals are performed with a scale independent gaussian.

$$P_l := S_{l/l_0}\, B_{[l_0,l[} := S_{l/l_0} \cdot \mu_{[l_0,l[}* \tag{2.87}$$

Some properties are valid only when $l_0 = 1$; we sometimes leave explicitly l_0 in equations where it is equal to 1 for clarity, and add cautionary remarks such as "provided $l_0 = 1$".

Properties of the coarse-graining operator.

- P_l obeys a multiplicative group property, provided $l_0 = 1$. Indeed

$$P_{l_2} P_{l_1} = P_{l_2 l_1/l_0}. \tag{2.88}$$

Proof

$$\begin{aligned} P_{l_2} P_{l_1} &= S_{l_2/l_0}\mu_{[l_0,l_2[} * \left(S_{l_1/l_0}\mu_{[l_0,l_1[}\right) * \\ &= S_{l_2/l_0} S_{l_1/l_0} \left(\mu_{l_0 l_1/l_0, l_2 l_1/l_0} * \mu_{[l_0,l_1[}\right) * \\ &= S_{\frac{l_2 l_1}{l_0}\frac{1}{l_0}}\, \mu_{\left[l_0, \frac{l_2 l_1}{l_0}\right[}* \end{aligned}$$

□

- P_l does not define a group because convolution does not have an inverse. Information is lost by convolution.
- Wick ordered monomials defined by (2.69) and in Appendix **??**ID are (pseudo) eigenfunctions of the coarse-graining operator.

$$\begin{aligned} &P_l \int_{M^D} \mathrm{dvol}(x)\ :\phi^n(x):_{[l_0,\infty[} \\ &= \left(\frac{l}{l_0}\right)^{n[\phi]+4} \int_{M^D} \mathrm{dvol}(x)\ :\phi^n(x):_{[l_0,\infty[} \end{aligned} \tag{2.89}$$

If the integral is over a finite volume, the volume is scaled down by S_{l/l_0}. Hence we use the expression pseudo-eigenfunction.

Proof

$$\begin{aligned} P_l\ :\phi^n(x):_{[l_0,\infty[} &= S_{l/l_0}\mu_{[l_0,l[} * \exp\Big(-\frac{\imath}{2}\Delta_{[l_0,l[}\phi^n(x)\Big) \\ &= S_{l/l_0} \exp\Big(\frac{\imath}{2}\Delta_{[l_0,l[} - \frac{\imath}{2}\Delta_{[l_0,\infty[}\Big)\,\phi^n(x) \\ &= S_{l/l_0} \exp\Big(-\frac{\imath}{2}\Delta_{[l,\infty[}\Big)\,\phi^n(x) \\ &= \exp\Big(-\frac{\imath}{2}\Delta_{[l_0,\infty[}\Big)\, S_{l/l_0}\phi^n(x) \end{aligned}$$

$$= \exp\Big(-\frac{\imath}{2}\Delta_{[l_0,\infty[}\Big)\left(\frac{l}{l_0}\right)^{n[\phi]}\phi^n\left(\frac{l_0}{l}x\right)$$
$$= \left(\frac{l}{l_0}\right)^{n[\phi]} : \phi^n\left(\frac{l_0}{l}x\right) :_{[l_0,\infty[} \tag{2.90}$$

Note that P_l preserves the scale range. Integrating over x both sides of (2.90) gives, after a change of variable $\frac{l_0}{l}x \mapsto x'$, eq. (2.89) — including the scaling down of integration when the domain is finite. □

- The coarse-graining operator satisfies a parabolic evolution equation in l.

$$\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right)P_l F(\phi) = 0 \tag{2.91}$$

where

$$\dot{S} = \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} S_{l/l_0} \quad \text{and} \quad \dot{\Delta} = \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} \Delta_{[l_0,l[}. \tag{2.92}$$

Explicitly

$$\dot{\Delta}(F(\phi)) = \int_{M^D} \mathrm{dvol}(x) \int_{M^D} \mathrm{dvol}(y) \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} G_{[l_0,l[}(|x-y|) \cdot \frac{\delta^2 F(\phi)}{\delta\phi(x)\,\delta\phi(y)} \tag{2.93}$$

with

$$\left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} G_{[l_0,l[}(\xi) = \left.\frac{\partial^\times}{\partial l}\right|_{l=l_0} \int_{l_0}^{l} d^\times s\, S_{s/l_0}\, u(\xi) = u(\xi)\,. \tag{2.94}$$

Note that u is independent of the scale.

Remark Frequently u is labelled $\dot{G}$, an abbreviation meaningful to the cognoscenti.

Proof One computes $\frac{\partial^\times}{\partial l}P_l$ at $l = l_0$, then one uses the semigroup property (2.88) valid for $l_0 = 1$ to prove the validity of the evolution equation (2.91) for all l when $l_0 = 1$. Starting from (2.87) the definition of P_l, one computes

$$\big(\mu_{[l_0,l[} * A\big)(\phi) = \int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi)\, A(\phi+\psi)\,. \tag{2.95}$$

The functional Taylor expansion of $A(\phi+\psi)$ up to second order is sufficient for deriving (2.91)

$$\begin{aligned}\left(\mu_{[l_0,l[} * A\right)(\phi) &= \\ &\int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi) \left(A(\phi) + \frac{1}{2}A''(\phi)\cdot\psi\psi + \cdots\right) && (2.96) \\ &= A(\phi)\int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi) + \frac{1}{2}\frac{s}{2\pi}\Delta_{[l_0,l[}A(\phi) && (2.97)\end{aligned}$$

where $\Delta_{[l_0,l[}A(\phi)$ is the functional Laplacian (2.65)

$$\Delta_{[l_0,l[} = \int \mathrm{dvol}(x)\int \mathrm{dvol}(y)\ G_{[l_0,l[}(x,y)\,\frac{\delta^2 A(\phi)}{\delta\phi(x)\,\delta\phi(y)} \tag{2.98}$$

obtained by the ψ integration in (2.96)

$$\int_{\mathbf{\Phi}} d\mu_{[l_0,l[}(\psi)\ \psi(x)\ \psi(y) = \frac{s}{2\pi}G_{[l_0,l[}(x,y)\,. \tag{2.99}$$

Finally

$$\left.\frac{\partial^\times}{\partial l}\right|_{l=l_0}\left(S_{l/l_0}\mu_{[l_0,l[} * A\right)(\phi) = \left(\dot{S} + \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right)A(\phi)\,. \tag{2.100}$$

□

- The generator H of the coarse graining operator is defined by

$$H := \left.\frac{\partial^\times}{\partial l}P_l\right|_{l=l_0}, \tag{2.101}$$

equivalently

$$P_l =: \exp\frac{l}{l_0}H. \tag{2.102}$$

The evolution operator (2.91) can therefore be written

$$\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta} = \frac{\partial^\times}{\partial l} - H. \tag{2.103}$$

The generator H operates on Wick monomials as follows (2.89)

$$\begin{aligned} H : \phi^n(x) :_{[l_0,\infty[} &= \left.\frac{\partial^\times}{\partial l}P_l : \phi^n(x) :\right|_{l=l_0} \\ &= \left.\frac{\partial^\times}{\partial l}\left(\frac{l}{l_0}\right)^{n[\phi]} : \phi^n\left(\frac{l_0}{l}x\right) :_{[l_0,\infty[}\right|_{l=l_0} && (2.104)\end{aligned}$$

The second order operator H, consisting of scaling and convolution, operates on Wick monomials as a first order operator.

- Coarse grained integrands in gaussian integrals

$$\left\langle \mu_{[l_0,\infty[}, A \right\rangle = \left\langle \mu_{[l_0,\infty[}, P_l A \right\rangle . \tag{2.105}$$

Proof

$$\begin{aligned} \left\langle \mu_{[l_0,\infty[}, A \right\rangle &= \left\langle \mu_{[l,\infty[}, \mu_{[l_0,l]} * A \right\rangle \\ &= \left\langle \mu_{[l_0,\infty[}, S_{l/l_0} \cdot \mu_{[l_0,l]} * A \right\rangle \\ &= \left\langle \mu_{[l_0,\infty[}, P_l A \right\rangle \end{aligned}$$

□

The important step in this proof is the second one,

$$\mu_{[l,\infty[} = \mu_{[l_0,\infty[} S_{l/l_0} ;$$

eq. (2.105) is used in section **??**16.1 for deriving the scale evolution of the effective action.