16

Renormalization 3: Scaling†

16.1 Renormalization group

> "The aim of the renormalization group is to describe how the dynamics of a system evolves as one changes the scale of the phenomena being observed."
> D. J. Gross [Gross]

> "The basic physical idea underlying the renormalization group is that many length or energy scales are locally coupled."
> K. J. Wilson [Wilson]

While working out estimates on renormalization group transformation, D. C. Brydges, J. Dimock and T. R. Hurd, developed, cleaned up, and simplified scaling techniques (see section 2.5, coarse-graining). In brief, a gaussian covariance can be decomposed into scale dependent contributions (2.79**??**). Indeed a gaussian $\mu_{s,G}$, abbreviated to μ_G, of covariance G and variance W, is defined by

$$\int_\Phi d\mu_G(\phi)\, \exp(-2\pi\imath \langle J,\phi\rangle) := \exp(-\pi s W(J))\,. \tag{16.1}$$

$$W(J) =: \langle J, GJ\rangle \tag{16.2}$$

It follows that if $W = W_1 + W_2$, the gaussian μ_G can be decomposed into two gaussians μ_{G_1}, μ_{G_2}:

$$\mu_G = \mu_{G_1} * \mu_{G_2} = \mu_{G_2} * \mu_{G_1}, \tag{16.3}$$

† The scaling properties established in section 2.5**??** are briefly summarized in this section to make it reasonably self-contained.

or more explicitly

$$\begin{aligned}
&\int_\Phi d\mu_G(\phi)\, \exp(-2\pi\imath\, \langle J, \phi\rangle) \\
&\quad = \int_\Phi d\mu_{G_2}(\phi_2) \int_\Phi d\mu_{G_1}(\phi_1)\, \exp(-2\pi\imath J\,(\phi_1 + \phi_2))
\end{aligned} \qquad (16.4)$$

where

$$G = G_1 + G_2 \qquad (16.5)$$

$$\phi = \phi_1 + \phi_2. \qquad (16.6)$$

The convolution property (16.3) and the additive properties (16.5) (16.6) make it possible to decompose a gaussian covariance into scale dependent contributions as follows.

- Introduce an independent scale variable $l \in \,]0, \infty[$. A gaussian covariance is a homogenous two-point function of $I\!\!M^D$ (a d-dimensional euclidean or minkowskian space)

$$\frac{s}{2\pi} G(|x-y|) = \int_{\mathbf{\Phi}} d\mu_G(\phi)\, \phi(x)\, \phi(y). \qquad (16.7)$$

  Therefore the covariance can be represented by an integral

$$G(\xi) = \int_0^\infty d^\times l\, S_l\, u(\xi) \quad \text{where} \quad d^\times l = dl/l, \quad \xi = |x-y| \qquad (16.8)$$

  for some function u. The scaling operator S_l is defined by (2.71**??**) — (2.73**??**)

$$S_l\, u(\xi) = l^{2[\phi]} u\,(\xi/l)\,, \quad [\phi] \text{ is the length dimension of } \phi. \qquad (16.9)$$

  Example: See (2.72**??**):

$$G(\xi) = c_D/\xi^{D-2}. \qquad (16.10)$$

- Break the domain of integration of the scaling variable l into subdomains $\left[2^j l_0, 2^{j+1} l_0\right[$

$$G(\xi) = \sum_{j=-\infty}^{+\infty} \int_{2^j l_0}^{2^{j+1} l_0} d^\times l\, S_l\, u(\xi) \qquad (16.11)$$

  and set

$$G(\xi) =: \sum_{j=-\infty}^{+\infty} G_j(l_0, \xi) \quad \text{abbreviated to } \sum_j G_j(\xi). \qquad (16.12)$$

The contributions G_j to the covariance G are selfsimilar. Indeed, set $l = \left(2^j l_0\right) k$, then

$$\begin{aligned} G_j(l_0, \xi) &= \left(2^j l_0\right)^{2[\phi]} \int_1^2 d^\times k \, k^{[2\phi]} \, u\left(\xi/2^j l_0 k\right) \\ &= \left(2^j l_0\right)^{2[\phi]} G_0\left(1, \xi/2^j l_0\right) \\ &= S_{2^j l_0} \, G_0(1, \xi) \, . \end{aligned} \tag{16.13}$$

See another equivalent formulation in section 2.5**??** eq. (2.82**??**).

The corresponding field decomposition (2.85**??**)

$$\phi = \sum_{j=-\infty}^{+\infty} \phi_j ,$$

also written

$$\phi(x) = \sum_j \phi_j(l_0, x) \, . \tag{16.14}$$

The subdomains $\left[2^j l_0, 2^{j+1} l_0\right[$ are exponentially increasing for $j \geq 0$, and the subdomains $\left[2^{j-1} l_0, 2^j l_0\right[$ are exponentially decreasing for $j \leq 0$. A scale dependent covariance defines a scale dependent gaussian, a scale dependent functional Laplacian (2.66**??**), scale dependent Bargmann-Segal and Wick transforms (2.69**??**, 2.70**??**).

Remark Brydges uses a scale variable $k \in [1, \infty[$. The domain decomposition

$$\sum_{j=-\infty}^{+\infty} \left[2^j l_0, 2^{j+1} l_0\right[$$

is then reorganized as

$$\sum_{j=-\infty}^{+\infty} = \sum_{j<0} + \sum_{j \geq 0} ,$$

and l_0 set equal to 1, i.e.

$$]0, \infty[= [0, 1[\cup [1, \infty[\tag{16.15}$$

with $k \in [1, \infty[$ and $k^{-1} \in [1, 0[$. Note that

$$\int_0^1 d^\times l \, S_l \, u(\xi) = \int_1^\infty d^\times k \, S_{k^{-1}} \, u(\xi). \tag{16.16}$$

Remark: Mellin transform Eq. (16.8) can be introduced as the Mellin transform $\widetilde{f}(\alpha)$ of a function $f(t)$ decreasing sufficiently rapidly at infinity

$$\begin{aligned}\widetilde{f}(\alpha) &:= \int_0^\infty d^\times t\, f(t) t^\alpha \\ &= a^\alpha \int_0^\infty d^\times l\, f\left(\frac{a}{l}\right) l^{-a} \quad \text{where } l = a/t.\end{aligned}$$

For example

$$\frac{1}{a^{2-\epsilon}} = \frac{1}{2} \int_0^\infty d^\times l\, \exp\bigl(-a^2/4l^2\bigr)\, l^{-2+\epsilon}. \tag{16.17}$$

Scale dependent gaussians can be used for defining effective actions: break the action functional S into a free (quadratic) term and a interacting term

$$S = \frac{1}{2} Q + S_{\text{int}}. \tag{16.18}$$

Here a free action is an action which can be decomposed into scale dependent contributions which are selfsimilar in the following sense. The quadratic form in (16.18) defines a differential operator D

$$Q(\phi) = \langle D\phi, \phi \rangle \,.$$

Provided the domain $\mathbf{\Phi}$ of ϕ is properly restricted (e.g. by the problem of interest) the operator D has a unique Green's function G

$$DG = 1 \tag{16.19}$$

which can be used as the covariance of the gaussian μ_G defined by (2.32**??**,2.33**??**). If D is a linear operator with constant coefficients, $G(x, y)$ is a function of $|x - y| =: \xi$ with scale decomposition (16.12)

$$G(\xi) = \sum_j G_j(l_0, \xi)\,. \tag{16.20}$$

The contributions G_j satisfy the selfsimilar condition (16.13). The corresponding action functional Q is a “free” action.

For constructing effective actions one can split the domain of the scale variable in two domains

$$[\Lambda, \infty[= [\Lambda, L[\,\cup\, [L, \infty[\tag{16.21}$$

where Λ is a short distance (high energy) cut off, used for handling divergences at the spacetime origin in euclidean quantum field theory, or on the lightcone at the origin in the minkowskian case. The variable L is the renormalization scale.

The quantity to be computed, formally written $\int \mathcal{D}\phi \, \exp\big(\frac{\imath}{\hbar} S(\phi)\big)$, is the limit $\Lambda = 0$ of

$$\begin{aligned} I_\Lambda &:= \int_{\mathbf{\Phi}} d\mu_{[\Lambda,\infty[}(\phi) \, \exp \frac{\imath}{\hbar} S_{\text{int}}(\phi) \\ &\equiv \left\langle \mu_{[\Lambda,\infty[}, \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle \end{aligned} \tag{16.22}$$

$$\left\langle \mu_{[\Lambda,\infty[}, \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle = \left\langle \mu_{[L,\infty[}, \mu_{[\Lambda,L[} * \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle . \tag{16.23}$$

The integrand on the right hand side is an effective action at scales greater than L, obtained by integrating short distance degrees of freedom in the range $[\Lambda, L[$. Eq. (16.23) transforms S_{int} into an effective action.

Eq. (16.23) is beautiful: an effective action at scales greater than L is integrated by a gaussian in the range $[L, \infty[$. But it is difficult to compute. It has been rewritten by Brydges et al. as a coarse-graining transformation (2.87**??**) with the coarse-graining operator P_l.

$$P_L := S_{l/l_0} \, \mu_{[l_0,l[} \tag{16.24}$$

where the scaling operator S_{l/l_0} is defined by (2.71**??**—2.73**??**). The coarse graining operator P_l is an element of a semigroup provided $l_0 = 1$ (2.88**??**). The semigroup property is necessary for deriving the scale evolution equation of the effective action. Therefore, henceforth $l_0 = 1$

$$P_l := S_l \, \mu_{[1,l[} \tag{16.25}$$

and the domain of the scale variable is split a $l = 1$

$$[\Lambda, \infty[= [\Lambda, 1[\cup [1, \infty[\,,$$

$$\left\langle \mu_{[\Lambda,\infty[}, \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle = \left\langle \mu_{[\Lambda,1[}, \mu_{[1,\infty[} * \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle . \tag{16.26}$$

The new integral can then be written as the scale independent gaussian of a coarse grained integrand (2.105**??**)

$$\begin{aligned}\left\langle \mu_{[1,\infty[}, \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle &= \left\langle \mu_{[l,\infty[}, \mu_{[l,l[} * \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle \\ &= \left\langle \mu_{[1,\infty[}, S_l \mu_{[1,l[} * \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle \\ &= \left\langle \mu_{[1,\infty[}, P_l \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle .\end{aligned}$$

Therefore

$$\left\langle \mu_{[\Lambda,\infty[}, \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle = \left\langle \mu_{[\Lambda,1[}, \mu_{[1,\infty[} * P_l \exp \frac{\imath}{\hbar} S_{\text{int}} \right\rangle . \tag{16.27}$$

The scale evolution of a coarse grained quantity $P_l A(\phi)$ is given by (2.92**??**)

$$\left(\frac{\partial^\times}{\partial l} - H \right) P_l A(\phi) = 0 \tag{16.28}$$

$$H = \dot{S} + \frac{1}{2} \frac{s}{2\pi} \dot{\Delta} \tag{16.29}$$

where a dot over a symbol stands for $\frac{\partial^\times}{\partial l}\big|_{l=l_0}$ and the functional laplacian Δ is defined by (2.56**??**).

We are interested in the scale evolution of $P_l \exp \frac{\imath}{\hbar} S_{\int}(\phi)$ that is in the evolution equation (16.28) when $A(\phi)$ is an exponential. Repeating the derivation (2.96**??**–2.99**??**), the only difference is the calculation of

$$\begin{aligned}&\int_\Phi d\mu_{[l_0,l[}(\psi) \left(\exp B(\phi)\right)'' \psi\psi \\ &= \quad \left(\exp B(\psi)\right)'' \frac{s}{2\pi} G_{[l_0,l[}; \\ &= \quad \int \text{dvol}x \int \text{dvol}y \frac{s}{2\pi} G_{[l_0,l[}\left(|x-y|\right) \\ &\qquad\qquad \times \frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \exp B(\phi);\end{aligned} \tag{16.30}$$

$$\frac{\delta^2}{\delta\phi(x)\delta\phi(y)} \exp B(\phi) = \left(\frac{\delta^2 B(\phi)}{\delta\phi(x)\delta\phi(y)} + \frac{\delta B(\phi)}{\delta\phi(x)} \frac{\delta B(\phi)}{\delta\phi(y)} \right) \exp B(\phi). \tag{16.31}$$

Finally $P_l B(\phi)$ satisfies the equation

$$\left(\frac{\partial^\times}{\partial l} - H\right) P_l B(\phi) = \frac{1}{2}\frac{s}{2\pi} B(\phi)\overset{\leftrightarrow}{\dot{\Delta}} B(\phi) \tag{16.32}$$

with

$$B(\phi)\overset{\leftrightarrow}{\dot{\Delta}} B(\phi) = \int \mathrm{dvol}x \int \mathrm{dvol}y G_{[l_0,l[}(|x-y|)\,\frac{\partial B(\phi)}{\partial\phi(x)}\frac{\partial B(\phi)}{\partial\phi(y)}. \tag{16.33}$$

Given (16.11)

$$G_{[l_0,l[}(\xi) = \int_{l_0}^{l} d^\times s\, S_{s/l_0}\, u(\xi), \tag{16.34}$$

$$\left.\frac{\partial^\times}{\partial l} G_{[l_0,l[}(\xi)\right|_{l=l_0} = u(\xi) \tag{16.35}$$

and

$$B(\phi)\overset{\leftrightarrow}{\dot{\Delta}} B(\phi) = \int \mathrm{dvol}x \int \mathrm{dvol}y\, u(|x-y|)\,\frac{\delta B(\phi)}{\delta\phi(x)}\frac{\delta B(\phi)}{\delta\phi(y)}. \tag{16.36}$$

In conclusion, we have derived two *exact* scale evolution equations (16.28) and (16.33); one for $P_l S_{\mathrm{int}}(\phi)$ and one for $P_l \exp\frac{\imath}{\hbar}S_{\mathrm{int}}(\phi)$. Set

$$S_{[l]} := P_l S_{\mathrm{int}}(\phi). \tag{16.37}$$

The coarse grained interaction $S_{[l]}$ (not to be confused with S_l) is often called the naive scaling of the interaction, or its scaling by engineering dimension, hence the use of the square bracket. $S_{[l]}$ satisfies (16.28)

$$\left(\frac{\partial^\times}{\partial l} - H\right) S_{[l]} = 0. \tag{16.38}$$

Let $S(l,\phi)$ be the effective action defined by

$$\exp S(l,\phi) := P_l \exp\frac{\imath}{\hbar}S_{\mathrm{int}}(\phi) \tag{16.39}$$

then

$$\left(\frac{\partial^\times}{\partial l} - H\right) S(l,\phi) = \frac{1}{2}\frac{s}{2\pi} S(l,\phi)\overset{\leftrightarrow}{\dot{\Delta}} S(l,\phi). \tag{16.40}$$

Approximate solutions to the exact scale evolution equation of the effective action are worked out in the next section for the $\lambda-\phi^4$ system. The approximate solutions include divergent terms. The strategy is *not* to drop divergent terms, nor to add counterterms by *fiat*, but to "trade" divergent terms for conditions on the scale dependent couplings; under the trade, the approximate solution remains valid at the same level of approximation. The trade requires the couplings to satisfy ordinary differential equations in l. These equations are equivalent to the renomalization flow equations.

16.2 The $\lambda - \phi^4$ system

The $\lambda - \phi^4$ system is a self-interacting relativistic scalar field in $1+3$ dimensions described by the Lagrangian density

$$\mathcal{L}(\phi(x)) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x) \tag{16.41}$$

$$\eta_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)\,. \tag{16.42}$$

λ is a dimensionless coupling constant when $\hbar = c = 1$.

There are many references for the study of the $\lambda - \phi^4$ system because it is the simplest nontrivial example without gauge fields. The chapter "Relativistic scalar field theory" is Ashok Das' book [Ashok Das] is an excellent introduction. In this section we apply to this system the scale evolution equation (16.40) for the effective action as developed by Brydges et al.

- One minor difference: Brydges uses $\hbar = c = 1$ and gives physical dimensions in mass dimension; we use $\hbar = c = 1$ and the length dimension because most of the work is done in spacetime rather than in four-momentum space. The relationship

$$l\frac{d}{dl} = -m\frac{d}{dm} \tag{16.43}$$

  provides the needed correspondence.
- A nontrivial difference: Brydges investigates euclidean quantum field theory, we investigate minkowskian field theory. The difference will appear in divergent expressions, singular at the origin in the euclidean case, singular on the lightcone in the minkowskian case.

A normal ordered lagrangian

The starting point of Brydges' work is the normal ordered Lagrangian

$$: \mathcal{L}(\phi(x)) : \;\; = \frac{1}{2}\eta^{\mu\nu} : \phi_{,\mu}(x)\phi_{,\nu}(x) : -\frac{1}{2}m^2 : \phi^2(x) : -\frac{\lambda}{4!} : \phi^4(x) : . \tag{16.44}$$

Originally *operator* normal ordering was introduced by Gian-Carlo Wick to simplify calculations (it readily identifies vanishing terms in vacuum expectation values). It also eliminates the (infinite) zero-point energy of an assembly of harmonic oscillators. The normal ordering of a *functional* F of ϕ is by definition

$$: F(\phi) :_G \equiv \exp\left(-\frac{1}{2}\Delta_G\right) F(\phi) \tag{16.45}$$

where Δ_G is the functional Laplacian defined by the covariance G

$$\Delta_G := \int d\mathrm{vol}x \int d\mathrm{vol}y \, G(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)}. \tag{16.46}$$

In quantum physics, a normal ordered Lagrangian in a functional integral corresponds to a normal ordered Hamiltonian operator in the corresponding matrix element (see Appendix ID**??**).

Normal ordered action functionals simplify calculations because integrals of normal ordered monomials are eigenvalues of the coarse-graining operator P_l (2.89**??**).

The effective action $S(l, \phi)$

We apply to the Lagrangian (16.41) the procedure developed in section 16.1 for constructing an effective action.

The functional integral (16.22) can be written in terms of (16.27) with an effective action $S(l, \phi)$ defined by

$$\exp S(l, \phi) := P_l \exp \frac{\imath}{\hbar} S_{\mathrm{int}}(\phi), \tag{16.47}$$

namely

$$\left\langle \mu_{[\Lambda,\infty[}, \exp \frac{\imath}{\hbar} S_{\mathrm{int}} \right\rangle = \left\langle \mu_{[\Lambda,1[}, \mu_{[1,\infty[} * P_l \frac{\imath}{\hbar} S_{\mathrm{int}}(\phi) \right\rangle . \tag{16.48}$$

The integral with respect to $\mu_{[1,\infty[}$ is over polynomial normal ordered accordingly by the covariance

$$G_{[1,\infty[}(|x-y|) = \int_{-1}^{\infty} d^{\times} l \, S_l \, u(|x-y|) \,. \tag{16.49}$$

Given an action

$$S(\phi) = \int \mathrm{dvol}x \mathcal{L}(\phi(x)) = \frac{1}{2} Q(x) + S_{\mathrm{int}}(\phi), \tag{16.50}$$

there often different ways of splitting it into a quadratic term and an interaction term. Choosing Q is choosing the concomitant gaussian μ_G. For the $\lambda - \phi^4$ system there are three natural choices for Q, namely

$$\begin{aligned} Q(x) &= \int \mathrm{dvol}x \, \eta^{\mu\nu} : \partial_\mu \phi(x) \partial_\nu \phi(x) : \quad \text{Brydges' choice} && (16.51) \\ Q(x) &= \int \mathrm{dvol}x \, \eta^{\mu\nu} : \partial_\mu \phi(x) \partial_\nu \phi(x) : -m^2 : \phi^2(x) : \\ Q(x) &= S''(\psi_{\mathrm{cl}}) \, \phi\phi, && (16.52) \end{aligned}$$

where $S''(\psi_{\mathrm{cl}})$ is the value of the second variation at a classical solution of the Euler-Lagrange equation. Brydges' choice leads to a mass and coupling constant renormalization flow equations. The second choice has not been worked out. The covariance $G(x,y)$ is, as in Brydges' choice, a function of $|x-y|$ but a more complicated one. The third choice (16.52) is challenging: the second variation as a quadratic form is very beneficial in the study of the anharmonic oscillator†.

We proceed with Brydges' choice (16.51). Therefore

$$S_{\mathrm{int}}(\phi) = \int \mathrm{dvol}x \left(-\frac{1}{2} m^2 : \phi^2(x) :_{[1,\infty[} - \frac{\lambda}{4!} : \phi^4(x) :_{[1,\infty[} \right). \tag{16.53}$$

The effective action $S(l,\phi)$ at scales larger than l defined by

$$P_L \exp \frac{\imath}{\hbar} S_{\mathrm{int}}(\phi) =: \exp S(l,\phi) \tag{16.54}$$

satisfies the exact scale evolution equation (16.40) $E(S)$

$$E(S) \equiv \left(\frac{d^{\times}}{dl} - \dot{S} - \frac{1}{2} \frac{s}{2\pi} \dot{\Delta} \right) S(l,\phi) - \frac{1}{2} \frac{s}{2\pi} S(l,\phi) \overset{\leftrightarrow}{\dot{\Delta}} S(l,\phi) = 0. \tag{16.55}$$

An approximation $T(l,\phi)$ to $S(l,\phi)$ is called a solution at order $\mathcal{O}\left(\vec{\lambda}^k\right)$

† A first step towards using the second variation of $S(\phi)$ for computing the effective action of the $\lambda - \phi^4$ system has been done by Xiaorong Wu Morrow; it is based on an effective action for the anharmonic oscillator. (unpublished)

if $E(T)$ is of order $\mathcal{O}\left(\vec{\lambda}^{k+1}\right)$. We use $\vec{\lambda}$ to designate the set of coupling constants (m, λ and possibly others). In order to present the key issues as simply as possible we consider a massless system, $m = 0$; henceforth

$$S_{\rm int}(\phi) = -\frac{\lambda}{4!}\int d{\rm vol}x \ :\phi^4(x):_{[1,\infty[} \ . \tag{16.56}$$

First order approximation to the effective action

The coarse grained interaction (naive scaling of the interaction)

$$S_{[l]}(\phi) = P_l S_{\rm int}(\phi) = -\frac{\lambda}{4!} l^{4+4[\phi]}\int d{\rm vol}x \ :\phi^4(x) := S_{\rm int}(\phi) \tag{16.57}$$

satisfies the evolution equation (16.38)

$$\left(\frac{d^\times}{dl} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) S_{[l]}(\phi) = 0. \tag{16.58}$$

$S_{[l]}$ is an approximate solution of order λ to $E(S)$ (16.55). The coupling constant λ is dimensionless and is not modified by the coarse graining operator P_l.

Second order approximation to the effective action

We expect the second order approximation $T(l,\phi)$ to $S(l,\phi)$ to have the following structure

$$T(l,\phi) = S_{[l]}(\phi) + \frac{1}{2} S_{[l]}(\phi) \overset{\leftrightarrow}{O} S_{[l]}(\phi), \tag{16.59}$$

the operator $\overset{\leftrightarrow}{O}$ being such that $E(T)$ be of order λ^3. The ansatz proposed by Brydges is

$$\overset{\leftrightarrow}{O} = \exp\frac{s}{2\pi}\overset{\leftrightarrow}{\Delta}_{[l^{-1},1[} -1. \tag{16.60}$$

In the case of the $\lambda - \phi^4$ system the exponetial terminates at $\overset{\leftrightarrow}{\Delta}^4$.

$$\begin{aligned}\overset{\leftrightarrow}{O} \ &= \ \frac{s}{2\pi}\overset{\leftrightarrow}{\Delta}_{[l^{-1},1[} + \frac{1}{2!}\left(\frac{s}{2\pi}\right)^2 \overset{\leftrightarrow}{\Delta}{}^2_{[l^{-1},1[} \\ &\quad + \frac{1}{3!}\left(\frac{s}{2\pi}\right)^3 \overset{\leftrightarrow}{\Delta}{}^3_{[l^{-1},1[} + \frac{1}{4!}\left(\frac{s}{2\pi}\right)^4 \overset{\leftrightarrow}{\Delta}{}^4_{[l^{-1},1[}\end{aligned} \tag{16.61}$$

Proof that, with the ansatz (16.60)

$$E(T) = \mathcal{O}\big(\lambda^3\big)\,. \tag{16.62}$$

A straight calculation of $E(T)$ is possible but far too long to be included here. We refer to the original literature [Brydges, Alex, Marcus], and outline the key features of the proof. We shall show that

$$\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) T(l,\phi) = \frac{1}{2} T(l,\phi) \frac{s}{2\pi} \dot{\overleftrightarrow{\Delta}} T(l,\phi) + \mathcal{O}\big(\lambda^3\big)\,. \tag{16.63}$$

$T(l,\phi)$ is the sum of a term of order λ, four terms of order λ^2:

$$\begin{aligned} T(l,\phi) \quad &= \quad S_{[l]}(\phi) + \frac{1}{2}\sum_{j=1}^{4} \frac{1}{j!}\left(\frac{s}{2\pi}\right)^j S_{[l]}(\phi)\, \overleftrightarrow{\Delta}^j_{[l^{-1},1[}\, S_{[l]}(\phi) \qquad (16.64) \\ &=: \quad S_{[l]}(\phi) + \sum_{j=1}^{4} T_j(l,\phi) \end{aligned}$$

- Computing the l.h.s. of (16.63) for $T_1(l,\phi)$

$$\begin{aligned} &\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) S_{[l]}(\phi)\, \overleftrightarrow{\Delta}_{[l^{-1},1[}\, S_{[l]}(\phi) \\ &\qquad = S_{[l]}(\phi)\left(\dot{\overleftrightarrow{\Delta}} - \frac{s}{2\pi}\dot{\overleftrightarrow{\Delta}}\,\overleftrightarrow{\Delta}_{[l^{-1},1[}\right) S_{[l]}(\phi). \end{aligned} \tag{16.65}$$

  At first sight, this result is unexpected, because it seems that the leibnitz product rule has been applied to the l.h.s. of (16.64) although it includes a scaling operator and a second order differential operator. A quick hand waving argument runs as follows: $\dot{S} + \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}$ is the gnerator H of the coarse grainingin operator. When acting on $S_{[l]}$ (or any Wick ordered monomials) it acts as a first order opeaotor (2.105**??**). So far so good, let us apply the leibnitz rule. Because $\left(\frac{\partial^\times}{\partial l} - H\right) S_{[l]}(\phi) = 0$ the only remaining term is

$$\begin{aligned} &\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) S_{[l]}\, \overleftrightarrow{\Delta}_{[l^{-1},1[}\, S_{[l]}(\phi) \\ &\qquad = S_{[l]}(\phi)\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) \overleftrightarrow{\Delta}_{[l^{-1},1[}\, S_{[l]}(\phi). \end{aligned} \tag{16.66}$$

  But, what is the meaning of the r.h.s.? How does the laplacian $\dot{\Delta}$

operate when bracketed between two functionals? How can the r.h.s. of (16.65) be equal to the r.h.s. of (16.64)? It is easy to prove that

$$\begin{aligned}\left(\frac{\partial^\times}{\partial l} - \dot{S}\right) G_{[l^{-1},1[}(\xi) &= \left(\frac{\partial^\times}{\partial l} - \dot{S}\right) \int_{l^{-1}}^{1} d^\times s\, S_s\, u(\xi) \\ &= S_{l^{-1}}\, u(\xi) - \left.\frac{\partial^\times}{\partial t}\right|_{t=1} \int_{l^{-1}t^{-1}}^{t^{-1}} d^\times s\, S_s\, u(\xi) \\ &= u(\xi).\end{aligned}$$

Therefore

$$\left(\frac{\partial^\times}{\partial l} - \dot{S}\right) \overleftrightarrow{\Delta}_{[l^{-1},1[} = \dot{\overleftrightarrow{\Delta}}. \tag{16.67}$$

But one cannot prove that $\frac{1}{2}\dot{\Delta}\ \overleftrightarrow{\Delta}_{[l^{-1},1[}$ is equal to $\dot{\overleftrightarrow{\Delta}}\ \overleftrightarrow{\Delta}_{[l^{-1},1[}$. The reason lies in the improper use of the leibnitz rule. Indeed

$$\frac{1}{2}\dot{\overleftrightarrow{\Delta}} S_{[l]}(\phi)\ \overleftrightarrow{\Delta}_{[l^{-1},1[}\ S_{[l]}(\phi)$$

contains in addition to $S_{[l]}(\phi)\dot{\overleftrightarrow{\Delta}}\ \overleftrightarrow{\Delta}_{[l^{-1},1[}\ S_{[l]}(\phi)$ terms proportional to the third variation of $S_{[l]}(\phi)$ which are canceled in the pedestrian computation of (16.64).

- Given (16.65) for T_1, we can record graphically the other terms necessary to prove that T satisfies (16.63). Let a straight line stand for $\overleftrightarrow{\Delta}_{[l^{-1},1[}$ and a dotted line for $\dot{\overleftrightarrow{\Delta}}$,

$$\begin{aligned}&\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right)\frac{s}{2\pi} S_{[l]}\text{———}S_{[l]} \\ &= \frac{s}{2\pi} S_{[l]}\ldots\ldots S_{[l]} - \left(\frac{s}{2\pi}\right)^2 S_{[l]}\overline{\ldots\ldots}S_{[l]}\end{aligned} \tag{16.68}$$

$$\begin{aligned}&\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right)\frac{1}{2}\left(\frac{s}{2\pi}\right)^2 S_{[l]}\overline{\overline{\qquad}}S_{[l]} \\ &= \left(\frac{s}{2\pi}\right)^2 S_{[l]}\overline{\ldots\ldots}S_{[l]} - \left(\frac{s}{2\pi}\right)^3 S_{[l]}\overline{\overline{\ldots\ldots}}S_{[l]}\end{aligned} \tag{16.69}$$

etc.

When adding up these equations the only remaining term on the r.h.s. is $\frac{s}{2\pi} S_{[l]}\ldots\ldots S_{[l]}$. Therefore

$$\left(\frac{\partial^\times}{\partial l} - \dot{S} - \frac{1}{2}\frac{s}{2\pi}\dot{\Delta}\right) S_{[l]}\ \overleftrightarrow{O}\ S_{[l]} = \frac{s}{2\pi} S_{[l]}\dot{\overleftrightarrow{\Delta}} S_{[l]} \tag{16.70}$$

and $T(l,\phi)$ satisfies (16.63). Brydges has proved that, in spite of the fact that the terms in $\mathcal{O}(\lambda^3)$ contain divergent terms when $l \to \infty$, they are uniformly bounded as $l \to \infty$.

The renormalization flow equation for λ

The approximate solution $T(l,\phi)$ of the effective action $S(l,\phi)$ contains one term of order λ and 4 terms of order λ^2 (16.64). We are interested in the divergent terms, they are the ones which will be "traded" for conditions on the couplings. The other terms are of no particular interest in renormalization. In the massless $\lambda\phi^4$ system there are two terms $T_2(l,\phi)$ and $T_3(l,\phi)$ which contain a singular part. As we shall see shortly only $T_2(l,\phi)$ contributes to the flow of λ. Explicitly

$$\begin{aligned} T_2(l,\phi) \quad &:= \quad \frac{1}{4}\left(\frac{s}{2\pi}\right)^2 \left(\frac{\lambda}{2}\right)^2 \int \mathrm{dvol}x \int \mathrm{dvol}y \\ &\qquad \times : \phi^2(x) : G^2_{[l^{-1},1[}(x-y) : \phi^2(y) : . \quad (16.71) \end{aligned}$$

We can write $T_2(l,\phi)$ as the sume of a regualr and a singular term by subtracting and adding to $T_2(l,\phi)$

$$\begin{aligned} T_{2\,\mathrm{sing}}(l,\phi) \quad &:= \quad \frac{1}{4}\left(\frac{s}{2\pi}\right)^2 \left(\frac{\lambda}{2}\right)^2 \int \mathrm{dvol}x \; : \phi^4(x) :_{[l^{-1},1[} \\ &\qquad \times \int \mathrm{dvol}y\, G^2_{[l^{-1},1[}(y). \quad (16.72) \end{aligned}$$

$T_{2\mathrm{sing}}$ is a local term which can be added to $S_{[l]}$, while it is subtracted from T_2 and regularizes it. Splitting $T_3(l,\phi)$ in a similar manner does not introduce terms including $: \phi^4(x) :$, and need not be considered in the λ-flow. The λ-flow equation consists in imposing to $S_{[l]} + T_{2\,\mathrm{sing}}$ the evolution equaiton (16.58) satisfied by $S_{[l]}$

$$\begin{aligned} \frac{1}{4!}\left(\frac{\partial^\times}{\partial l} - H\right)&\left(\lambda(l) + 36\left(\frac{s}{2\pi}\right)^2 \lambda^2(l) \int \mathrm{dvol}y\, G^2_{[l^{-1},1[}(y)\right) \\ &\times \int \mathrm{dvol}x \; : \phi^4(x) :_{[l]} = 0. \quad (16.73) \end{aligned}$$

The irrelevant factor $1/4!$ is included for the convenience of the reader who may have used λ rather than $\lambda/4!$ in the interaction term. For the same reason we leave $s/2\pi$ explicitly because it originates from the gaussian normalization (2.36**??**).

The λ-flow equation reduces to

$$\frac{\partial^\times}{\partial l}\left(\lambda(l) + 36\left(\frac{s}{2\pi}\right)^2 \lambda^2(l)\right) \int \mathrm{dvol}x\, G^2_{[l^{-1},1[}(x) = 0. \quad (16.74)$$

Proof H sees only $\int \mathrm{dvol}x\ :\phi^4(x):$ since the first factor does not depend on ϕ and is dimensionless, and

$$\left(\frac{\partial^\times}{\partial l} - H\right) \int \mathrm{dvol}x\ :\phi^4(x) := 0.$$

□

The flow equation (16.74) plays the role of the β-function derived by loop expansion in perturbative Quantum Field Theory.

$$\beta(\lambda) = -\frac{d^\times}{dl}\lambda(l) = \frac{3\lambda^2}{16\pi^2} \tag{16.75}$$

The β-function is constructed from an effective action at a given order of loops which is built with one-particle-irreducible diagrams. This method is efficient because the set of one-particle-irreducible diagrams is a much smaller set than the set of all possible diagrams for the given system. Within perturbation theory, there is strong indication [Marcus] (one could nearly say a proof) that both flow equations (16.74) and (16.75) give the same approximation to the flow equation.

The goal of this application is only to introduce scaling renormalization. We refer to the original publications for an in depth study of scaling renormalization† [Brydges] and to textbooks in quantum Field Theory for the use of β-functions.

The scaling method provides an exact equation (16.40) for the effective action $S(l, \phi)$. It has been used by Brydges, Dimock and Hurd for computing fixed points of the flow equation (16.34). In the euclidean case they found that the gaussian fixed point is unstable but that in dimension $4 - \epsilon$ there is a hyperbolic non-gaussian fixed point a distance $\mathcal{O}(\epsilon)$ away. In a neighborhood of this fixed point they constructed the stable manifold. Brydges et al. work with euclidean fields. His method has been applied to minkowskian fields by [Wurm, Berg].

† Brydges uses dimensional regularization by changing the dimension of the field rather than the space time dimension. In his work

$$\begin{aligned} 2[\phi] &= -2 + \epsilon \\ \text{and } [\lambda] &= 2\epsilon. \end{aligned}$$

Either dimensional regularization leads to the same final result. Brydges' method works on the action functional, rather than on the diagrams.