1

A Legacy

1.1 The beginning

In 1933 Dirac[2] laid the foundation stone for what was destined to become in the hands of Feynman a new formulation of quantum mechanics. Dirac introduced the action principle in quantum physics [2, 3], and Feynman, in his doctoral thesis [4] "The Principle of Least Action in Quantum Physics", introduced path integrals driven by the classical action *functional*, integral of the lagrangian. The power of this formalism was vindicated[5] when Feynman developed techniques for computing functional integrals and applied them to the relativistic computation of the Lamb shift.

Functional integration in mathematics was relatively little studied until Wiener's analysis[6] of "Differential Space"(1923). Wiener uses the term "differential space" to emphasize the advantage of working not with the values of a function but with the difference of two consecutive values. He constructed a measure in "Differential Space". Mark Kac remarked that the Wiener integral

$$\mathbb{E}\left[\exp\left(-\int_{\tau_a}^{\tau_b} V(q(\tau))\,d\tau\right)\right], \tag{1.1}$$

where $\mathbb{E}$ denotes the expectation value relative to the Wiener measure, becomes the Feynman integral

$$\int \mathcal{D}q \exp\, i\,S(q) \quad \text{for } S(q) = \int_{t_a}^{t_b} dt\left(\frac{1}{2}\dot{q}(t)^2 - V(q(t))\right) \tag{1.2}$$

* Because, says N.G. Van Kampen "When dealing with less simple and concrete equations, physical intuition is less reliable and often borders on wishful thinking".

if one sets $t = i\tau$. Kac concluded that because of i in the exponent, Feynman's theory is not easily made rigorous. It is possible[7][8] [see Technical Appendix A] to make sense of Kac's integral for $t = i\tau$. Feynman, however, objected to a Kac-type integral because "it spoils the physical unification of kinetic and potential parts of the action."[9] The kinetic contribution is hidden from sight.

The arguments of the functionals considered above are functions on a line. The line need not be a time line, it can be a scale line, a one parameter subgroup, etc... In all cases, the line can be used to "time" order products of values of the function at different times†. Given a product of factors $U_1, ..., U_N$ each attached to a point t_i on a line, one denotes by $T(U_1, \ldots, U_N)$ the product $U_{i_N} \ldots U_{i_1}$ where the permutation $i_1 \ldots i_N$ of $1 \ldots N$ is such that $t_{i_1} < \ldots < t_{i_N}$. Hence in the rearranged product the times attached to the factors increase from right to left.

The evolution of a system is dictated by the "time" line. Thus Dirac and Feynman expressed the time evolution of a system by the following N-tuple integral over the variables of integration $\{q'_i\}$ where q'_i is the "position" of the system at "time" t_i; in Feynman's notation the probability amplitude $(q'_t|q'_0)$ for finding at time t in position q'_t a system known to be at time t_0 in position q'_0 is given by

$$(q'_t|q'_0) = \int\int ... \int (q'_t|q'_N)\, dq'_N (q'_N|q'_{N-1}) dq'_{N-1} ... (q'_2|q'_1) dq'_1 (q'_1|q'_0). \tag{1.3}$$

The continuum limit, if it exists, is a "path integral" with domain of integration consisting of functions on $[t_0, t]$. Dirac showed that $(q'_t|q'_0)$ defines the exponential of a function $\mathcal{S}$,

$$\exp\left(i\, \mathcal{S}(q'_t, q'_o, t)/\hbar\right) := (q'_t|q'_0); \tag{1.4}$$

the function $\mathcal{S}$ is called by Dirac[3] the "quantum analogue of the classical action function (a.k.a. Hamilton's principal function)" because its real part is equal to the classical action function $\mathcal{S}$ and its imaginary part is of order $\hbar$. Feynman remarked that $(q'_{t+\delta t}|q'_t)$ is "often equal to

† There are many presentations of time ordering. A convenient one for our purpose can be found in *Mathemagics* [10]. In early publications, there are sometimes two different time ordering symbols: T^* which commutes with both differentiation and integration, and T which does not. T^* is the good time ordering and simply called T nowadays. The reverse time ordering defined by 1.3 is sometimes labeled $\overleftarrow{T}$

$$\exp \frac{i}{\hbar} L \left(\frac{q'_{t+\delta t} - q'_t}{\delta t}, q'_{t+\delta t} \right) \delta t \tag{1.5}$$

within a normalization constant in the limit as δt approaches zero."[4] Feynman expressed the finite probability amplitude $(q'_t|q'_0)$ as a path integral.

$$(q'_t|q'_0) = \int \mathcal{D}q \, \exp \frac{i}{\hbar} S(q) \tag{1.6}$$

where the action functional

$$S(q) = \int_{t_o}^{t} ds \, L\left(\dot{q}(s), q(s)\right). \tag{1.7}$$

The path integral (1.7) constructed from the infinitesimals (1.5) is a product integral (see Technical Appendix B for the definition, properties and references of product integrals). *The action functional, broken up in time steps, is a key building block of the path integral.*

Notation The Dirac quantum analogue of the classical action, labelled $\mathcal{S}$ will not be used. The action function, solution of the Hamilton-Jacobi equation, is labelled $\mathcal{S}$, and the action functional integral of the lagrangian is labelled S. The letters $\mathcal{S}$ and S are not strikingly different but clearly identified by the context. See Appendix E.

Two phrases give a foretaste of a rich and complex issue of path integrals: "the imaginary part [of $\mathcal{S}$] is of order $\hbar$" says Dirac, "within a normalization constant"† says Feynman who summarizes the issue in the symbol $\mathcal{D}q$.

Feynman rules for computing asymptotic expansions of path integrals, order by order in perturbation theory, are stated in terms of graphs, also called diagrams. The Feynman-diagram-technique is widely used because the diagrams are not only a powerful aid to the calculation but also an easy bookkeeping for the physical process of interest. Moreover the diagram expansion of functional integrals in quantum field theory proceeds like the diagram expansion of path integrals in quantum mechanics. The time ordering (1.3) becomes a chronological ordering, dictated by light cones: if the point x_j is in the future light cone of x_i, one

† See Chapter 2 for a brief comment on the early calculations of the normalization constant

writes $j \gg i$, and defines the chronological ordering T by the symmetric function

$$T\left(U_j\, U_i\right) = T\left(U_i\, U_j\right) := U_j\, U_i \quad \text{where} \quad U_i := U\left(x_i\right) \quad \text{and} \quad j \gg i. \tag{1.8}$$

The Feynman diagrams are a graphic expression of gaussian integrals of polynomials. The first step for computing the diagram expansion of a given functional integral is the expansion into polynomials of the exponential in the integrand. We shall give an explicit diagram expansion as an application of gaussian path integrals in section I.3.3.

1.2 Integrals over function spaces

Wiener and Feynman introduced path integrals as limit for $N = \infty$ of an N-tuple integral. Feynman noted that the limit of N-tuple integrals is at best a crude way for defining path integrals. Indeed, the drawbacks are several:

- How does one choose the short time probability amplitude $(q'_{t+\delta t}|q'_t)$ and the undefined normalization constant?
- How does one compute the N-tuple integral?
- How does one know if it has a unique limit for $N = \infty$?

The answer is to do away with N-tuple integrals and to identify the function spaces which serve as domains of integration for functional integrals. The theory of promeasures[11] (projective system of measures on topological vector spaces, locally convex, but not necessarily locally compact), combined with Schwartz distributions, yields a practical method for integrating on function spaces. The step from promeasures to prodistributions (introduced first as pseudomeasures[7]) is straightforward. An improved version is presented in section I.2. Already in its original form it has been used for computing nontrivial examples, e.g. the explicit cross section for glory scattering of waves by Schwarzschild black holes.[12]

1.3 Operator formalism

A functional integral is a mathematical object, but historically its use in physics is intimately connected with quantum physics. Matrix elements of an operator on Hilbert spaces or on Fock spaces have been used for defining their functional integral representation.

Bryce DeWitt[13] constructs the operator formalism of quantum physics from the Peierls bracket which leads to the Schwinger variational principle and to functional integral representations.

The bracket invented by Peierls[14] in 1952 is a beautiful, but often neglected, covariant replacement for the canonical Poisson bracket, or its generalizations, used in canonical quantization. Let A and B be any two physical observables. Their *Peierls bracket* (A, B) is by definition

$$(A, B) := \mathcal{D}_A^- B - (-1)^{\tilde{A}\tilde{B}} \mathcal{D}_B^- A \tag{1.9}$$

where the symbol $\tilde{A} \in \{0, 1\}$ is the Grassmann parity of A, and where $\mathcal{D}_A^- B$ $(\mathcal{D}_A^+ B)$ is known as the retarded (advanced) effect of A on B. The precise definition follows from the theory of measurement.

The *operator quantization rule* associates an operator $\mathbf{A}$ to an observable A; the supercommutator $[\mathbf{A},\mathbf{B}]$ is given by the Peierls bracket:

$$[\mathbf{A}, \mathbf{B}] = i\hbar (A, B) + O(\hbar^2). \tag{1.10}$$

Let $|A>$ be an eigenvector of the operator $\mathbf{A}$ for the eigenvalue A. The *Schwinger variational principle* states that the variation of the transition amplitude $< A|B >$ generated by the variation $\delta\mathbf{S}$ of an action $\mathbf{S}$, functional of field operators, acting on a space of state vectors is:

$$\delta < A|B >= i < A\, |\delta S/\hbar|\, B > \tag{1.11}$$

The variation of matrix elements have led to their functional integral representation. The solution of this equation obtained by Bryce DeWitt is the Feynman functional integral representation of $< A|B >$. It brings out explicitly the exponential of the classical action functional in the integrand, and the "measure" on the space of paths, or the space of histories, as the case may be. The domain of validity of this solution encompasses many different functional integrals needed in Quantum Field Theory and Quantum Mechanics. The Schwinger-DeWitt approach will be compared at appropriate places with the new approaches presented in the following sections. The measure, called $\mu(\phi)$, is an important contribution in the applications of functional integrals over fields ϕ. (See section 14.1).

1.4 A few titles

By now functional integration has proved itself. It is no more a "secret weapon used by a small group of mathematical physicists"† but is still not infrequently a "sacramental formula"‡. A bibliography of functional integration in physics, other than a computer generated list of references, would be an enormous task, best undertaken by a historian of science. We shall mention only a few books which together give an idea of the scope of the subject. In chronological order:

• R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1961)

• S.A. Albeverio and R.J. Hoegh-Krohn, *Mathematical Theory of Feynman Path Integrals*, Lecture Notes in Mathematics no. 523 (Springer-Verlag, 1976)

• Barry Simon, *Functional Integration and Quantum Physics* (Academic Press, New York, 1979)

• L.S. Schulman, *Techniques and Applications of Path Integration* (John Wiley, 1981)

• K.D. Elworthy, *Stochastic Differential Equations on Manifolds* (Cambridge University Press, 1982)

• Ashok Das, *Field Theory, a Path Integral Approach* (World Scientific, Singapore, 1993)

• Hagen Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics* (2nd edition) (World Scientific, Singapore, 1995)

• C. Grosche and F. Steiner, *Table of Feynman Path Integrals* (Springer Tracts in Modern Physics, 1995)

• M. Chaichian and A. Demichev, *Path Integrals in Physics* Vol I and II (IOP Bristol UK, 2001)

• G.W. Johnson and M.L. Lapidus, *The Feynman Integral and Feynman's Operational Calculus* (Oxford University Press - paperback 2002, first published in 2000)

• Bryce DeWitt, *The Global Approach to Quantum Field Theory* (Oxford University Press, 2003)

Many books on Quantum Field Theory include several chapters on

† "an extremely powerful tool used as a kind of secret weapon by a small group of mathematical physicists" B. Simon(1979).

‡ "A starting point of many modern works in various areas of theoretical physics is the path integral $\int \mathcal{D}q \exp i\, S(q)/\hbar$. What is the meaning of this sacramental formula?" M. Marinov (1991).

functional integration.

These few books, together with their bibliography, give a good picture of functional integration in physics at the beginning of the twenty first century. We apologize for an incomplete list of our legacy.

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