Torelli locus and Newton polygons

1.1

Families of Abelian varieties

arithmetic	geometry
- Jacobians	moduli spaces
- cyclic action	

Broad perspectives

open questions

Examples

new invariants: Newton polygons

$$\mathbb{C} \rightsquigarrow \overline{\mathbb{F}}_p$$

1.2 X p.p. abelian variety
dim g

today Examples $g=2$

$$X \cong \mathbb{C}^2/\Lambda$$ ← lattice

basis

$$\left[\, {}_{*}Z^{*} \;\middle|\; \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right]$$ period matrix

X algebraic

Riemann bilinear relation

$$Z = Z^T$$

<u>Lemma</u> A_2 moduli space of p.p. abelian var dimension 2

dim 3 irred

<u>Thm</u> Siegal

A_g moduli space of p.p. abelian varieties dim g

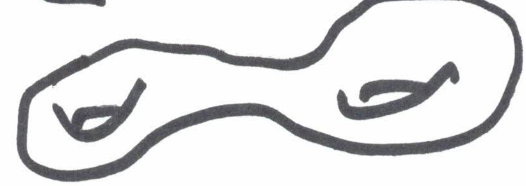
irred. dim $g(g+1)/2$

1.3 Ex. C curve of genus 2

Riemann-Roch: C hyperelliptic

$\pi : C \to \mathbb{P}^1$

degree 2



$C: \ y^2 = h(x)$

Riemann-Hurwitz: π has 6 branch points.

WLOG $\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\}$

$h(x) = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$

Lemma $\mathcal{M}_2$ moduli space smooth genus 2 curves

irred dim 3

Same dim as $\mathcal{A}_2$!

Thm $\mathcal{M}_g$ moduli space smooth curves genus g

irred dim $3g-3$ for $g \geq 2$.

$$\begin{array}{ccc} \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \\ C & \longrightarrow & \mathrm{Jac}(C) \end{array}$$

curve genus g — p.p. abel. var dim g

Ex. $g=2$

every point on $\mathrm{Jac}(C)$ can be written as $Q_1 + Q_2 - 2Q_0$ for Q_0 fixed, Q_1, Q_2 vary.

Torelli: Thm

If $C_1 \not\cong C_2$ not isom

then $Jac(C_1) \not\cong Jac(C_2)$

not isom.

$$T : \mathcal{M}_g \longrightarrow \mathcal{A}_g$$

injective on points.

$g=2$ same dim almost every abelian surface is a Jac (or $E_1 \oplus E_2$)

$g=3$ same dim

same idea.

$g \geq 4$ no longer true: most abel. var. not Jacobians.

1.5 / $\mathbb{C}$ ~~most~~ abelian varieties is it a Jacobian

open Q: Ekedahl-Serre

$g \geq 2$, does there exist a smooth curve C of genus g s.t. $\mathrm{Jac}(C) \sim E_1 \oplus \dots \oplus E_g$

isogenous

g elliptic curves

many cases: yes!

ES $g = 1297$.

recently: LMFDB $g = 38$ yes!

open case: $g = 59$, then $g = 66$ unknown.

1.6 Char $g=1$

$k=\bar{k}$ $\quad$ $\mathrm{char}(k)=p$

E elliptic curve $/k$

Def: ① E supersingular if $\mathrm{End}(\bar{E})$ not. comm.

Equivalently:

② $\#E[p](k) = \begin{cases} p & E \text{ ord} \\ 1 & E \text{ ss.} \end{cases}$

p-torsion

③ $E/\mathbb{F}_q$ $\quad$ $\#E(\mathbb{F}_q) = q+1-a$

$q=p^{\cancel{\cdot}}$ $\quad$ E ss $\Longleftrightarrow p \mid a$

1.6/ $E/\mathbb{F}_q$

L-poly.

zeta function

$$Z(E/\mathbb{F}_q, T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}$$

Newton polygon

E ss $\Longleftrightarrow$ Newton polygon slopes of 1/2

Thm: Deuring:

1.6 cont

<u>Thm</u> Deuring: for every p, there exists E supersingular (defined over $\mathbb{F}_{p^2}$)

$\# \lambda = \frac{p-1}{2}$

$\# ss \sim \frac{p-1}{12}$

Igusa:

$$y^2 = x(x-1)(x-\lambda)$$

1-form $\frac{dx}{y}$ Cartier operator

$$C\left(\frac{dx}{y}\right) = f(\lambda, p)\frac{dx}{y}$$

~~#~~ λ root of ~~[illegible]~~ iff E supersingular.

1.7. $g \geq 2$

X p.p. abel. var. dim g

Def $X/\overline{\mathbb{F}_p}$ supersingular means $X \sim E_1 \oplus \ldots \oplus E_g$

isogenous

$\overline{\mathbb{F}_p}$

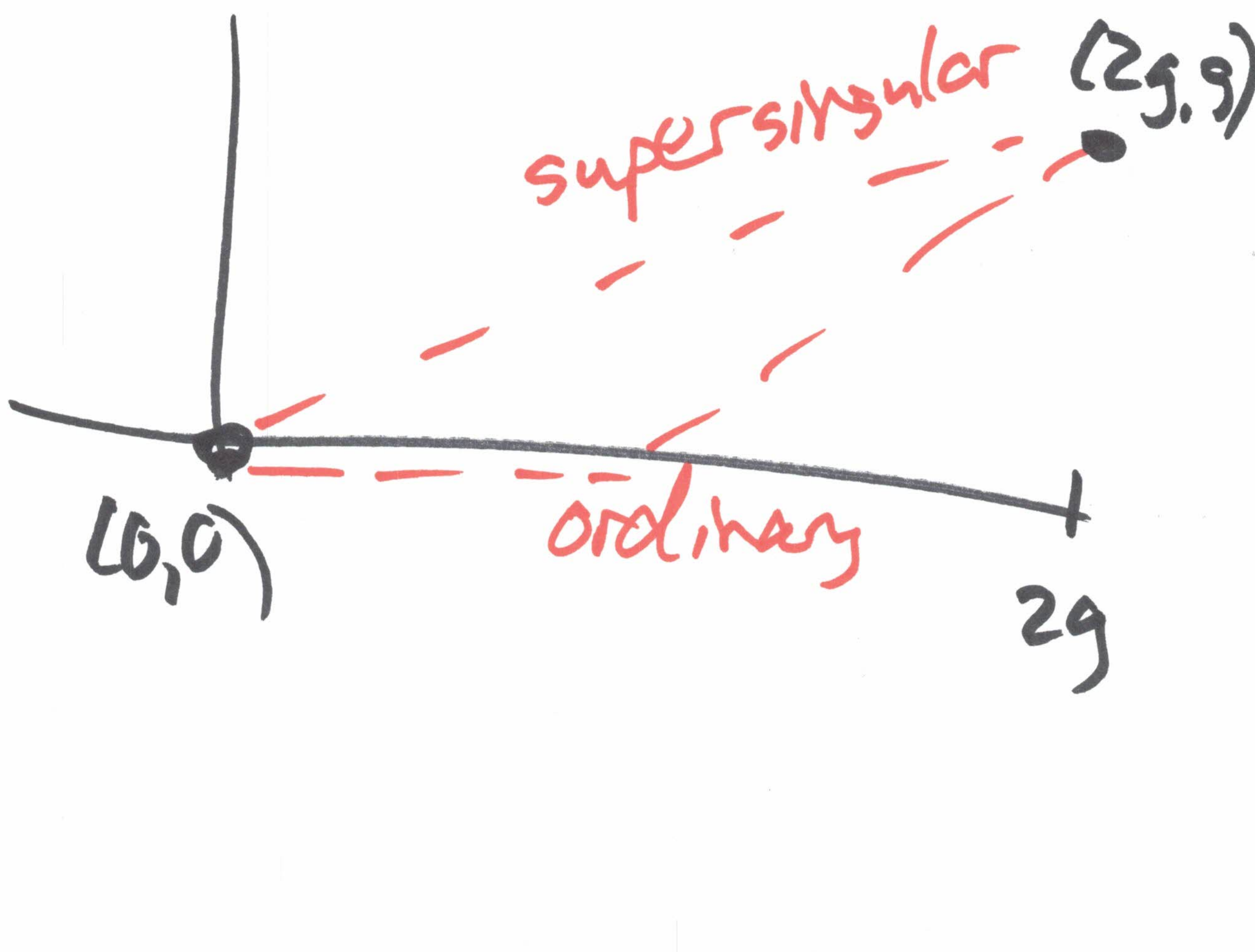
all supersing elliptic curves

Open question:

Given $g \geq 2$, given p prime. Does there exist $C/\overline{\mathbb{F}_p}$ smooth of genus g s.t. $\mathrm{Jac}(C)$ is supersingular?

1.8 ~~$\mathbb{A}$~~ $C/\mathbb{F}_q$

Newton poly of $L(C/\mathbb{F}_q, T)$
line segment of slope $1/2$



Does there exist supersingular smooth curve of genus g over $\overline{\mathbb{F}}_p$?

yes: $p=2$ for all g
Van der Geer + Vlugt.

<u>yes</u>: $g=2$ Serre for all p
$g=3$ Oort for all p
$g=4$ Kudo/Harashita/Senda

1.9 ~~p~~ $\forall$ primes p ~~(odd)~~ $p > 5$

Theorem There exists a smooth curve C of genus 2 that is supersingular

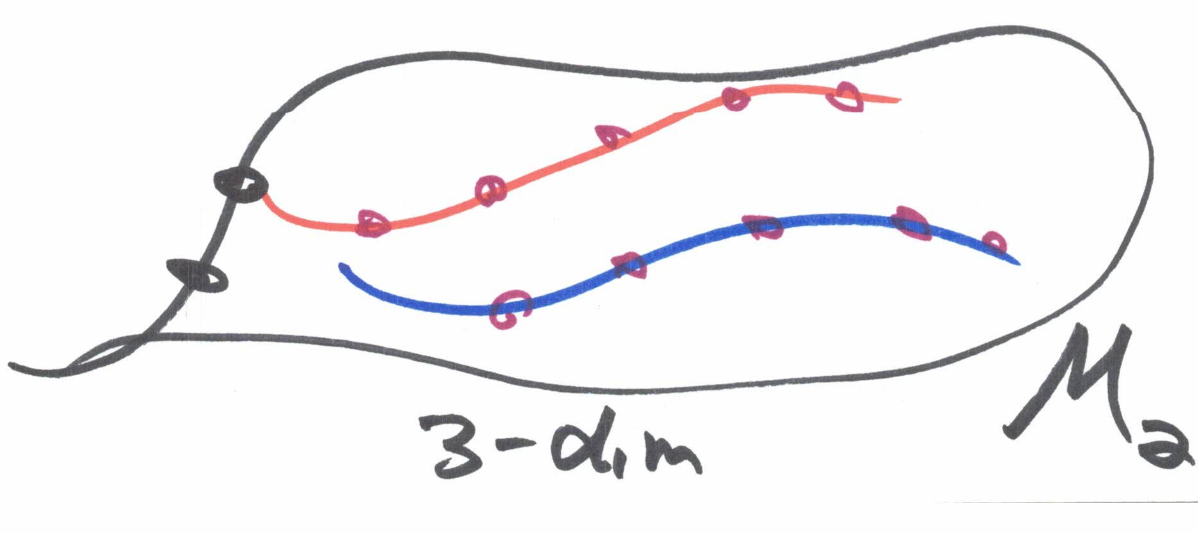
proof: Ibukiyama Katsura + Oort

(i) $y^2 = (x^3 - 1)(x^3 - t)$

$S_3 \subset \mathrm{Aut}(C)$

(ii) $y^2 = x(x^2 - 1)(x^2 - t)$

$D_4 \subset \mathrm{Aut}(C)$



IKO

$\#\left\{\begin{matrix}\text{ss curves} \\ \text{of } 2 \\ \text{in family}\end{matrix}\right\} \sim$ $\frac{p-1}{6}$ family (i)

$\frac{p-1}{8}$ family (ii)

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ E_1 & & E_2 \end{tikzcd}$$

$J_C \sim E_1 \oplus E_2$

E_1 ss $\Longleftrightarrow E_2$ ss

Cartier operator on $H^0(C, \Omega^1)$

$f(t,p)\begin{bmatrix} * & \\ & * \end{bmatrix}$

or $\begin{bmatrix} & * \\ * & \end{bmatrix}$

1972

Miller:

$$y^2 = x^{2g+1} + t x^{g+1} + x \quad \text{if } p \nmid g$$

$$y^2 = x^{2g+2} + t x^{2g+1} + 1 \quad \text{if } p \mid g$$

ordinary for typical t

not ordinary for some t