Algebraic Cycles on AV

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$/k$, X, Y : sm. proj. var. $/k$

Def: $i, j \in \mathbb{Z}$

$Z^i(X) := \mathbb{Z} \cdot \left\{ \begin{array}{c} \text{cl. irred subvar } Z \subset X \\ \text{of codim} = i \end{array} \right\}$

$Z_j(X) := \mathbb{Z} \cdot \left\{ \begin{array}{c} \text{———} \\ \dim(Z) = j \end{array} \right\}$

$\quad \| $

$Z^{\dim(X)-j}(X)$

elt: $\sum m_i Z_i \qquad m_i \in \mathbb{Z}$

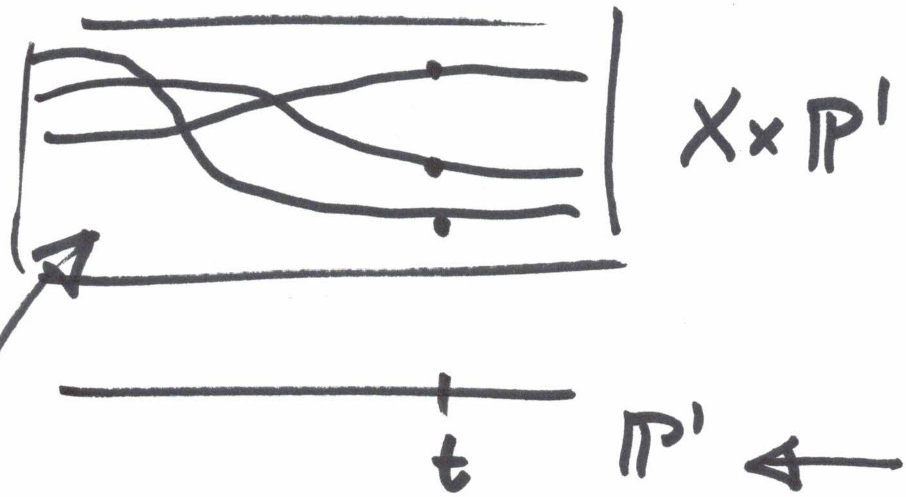
$\qquad Z_i$

Rat'l eq : $W \subset X$ irred of codim $= i-1$

$\downarrow$

gen'd by : $0 \neq f \in k(W) \rightsquigarrow \mathrm{div}(f)$

$\mathrm{div}(f) \sim_{rat} 0$



flat fam. $\mathcal{V}$ of cycles on $X \times \mathbb{P}^1$

at t : $\mathcal{V}_t$

Rat'l eq : gen'd by

$$\mathcal{V}_{t_1} \sim_{rat} \mathcal{V}_{t_2}$$

$$CH^i(X) := Z^i(X)/\sim_{rat}$$

$$CH_j(X) := Z_j(X)/\sim_{rat}$$

$$CH(X) := \bigoplus_i CH^i(X) = \bigoplus_j CH_j(X)$$

$$CH(X)_{\mathbb{Q}} := CH(X) \otimes \mathbb{Q}$$

Exa (X irred)

$$CH^0(X) \cong \mathbb{Z}.[X]$$

$$CH^1(X) = Cl(X) \cong Pic(X)$$

Operations

push-forward $f : X \longrightarrow Y \rightsquigarrow$

$$f_* : CH_*(X) \longrightarrow CH_*(Y)$$

Idea: $Z \subset X \rightsquigarrow f(Z) \subset Y$

$Z \longrightarrow f(Z)$: if gen. fin. of deg $= d$

then $f_*[Z] = d \cdot [f(Z)]$

else $f_*[Z] = 0$.

Pullback (Gysin) $f: X \to Y$

$\rightsquigarrow f^*: CH^*(Y) \longrightarrow CH^*(X)$

preserves codim-grading

Special case: f flat then for $W \subset Y$

$\rightsquigarrow f^{-1}(W) \rightsquigarrow f^*[W] = [f^{-1}(W)]$.

Intersection product: (X/k sm. proj)

$CH^*(X)$ is comm. graded ring

$\cdot : CH^i(X) \times CH^j(X) \to CH^{i+j}(X)$

Very special case: $W, Z \subset X$ inters. transversally:

$$[W] \cdot [Z] = [Z \cap W].$$

Exterior prod : $X, Y / k$

$$\left.\begin{array}{l} \alpha \in CH^i(X), \\ \beta \in CH^j(Y) \end{array}\right\} \rightsquigarrow \alpha \times \beta \in CH^{i+j}(X \times Y)$$

Idea : $\alpha = [W]$, $\beta = [Z]$ then $\alpha \times \beta = [W \times Z]$

Relations :

- $\alpha \cdot \beta = \Delta^*(\alpha \times \beta)$

  $\Delta : X \longrightarrow X \times X$

- $\alpha \times \beta = pr_1^*(\alpha) \cdot pr_2^*(\beta)$

$$\begin{tikzcd} & X \times X \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X & & X \end{tikzcd}$$

<u>Proj. formula</u> : $f : X \longrightarrow Y$

$$f_*\big(f^*(\alpha) \cdot \beta \big) = \alpha \cdot f_*(\beta)$$

X/k ab. var., $\dim = g$,

$$m : X \times X \longrightarrow X$$

$\boxed{!}$ $CH(X)$ has a 2^{nd} ring structure!

$$\star : CH_i(X) \times CH_j(X) \longrightarrow CH_{i+j}(X)$$

$$\alpha, \beta \longmapsto m_*(\alpha \times \beta)$$

$CH(X)$: Comm. graded ring

↑ for dim of cycles

$$X \xrightarrow{\Delta} X \times X \xrightarrow{m} X$$

$$\alpha \times \beta$$

$\alpha = [W]$, $\beta = [Z]$ $\rightsquigarrow$

$(W+Z) \subset X$

$\quad \parallel$

$\{P+Q \mid P \in W ,\ Q \in Z\}$

$W \times Z \xrightarrow{m} (W+Z)$ if gen. fin of degree $= d$ then $\alpha \star \beta = d \cdot [(W+Z)]$

unit for $\star$-prod $= [e]$

$X \rightsquigarrow X^t := \mathrm{Pic}^0_{X/k}$

$\mathrm{Pic}_{X/k}$ = moduli of line bun on X

$\cup$

Pic^0 comp. $\ni [\mathcal{O}_X]$.

Always : $X \sim X^t$

in gen'l : $X \not\cong X^t$.

Poincare LB : $\mathcal{P}$ on $X \times X^t$

$\xi \in X^t$: $\mathcal{P}|_{X \times \{\xi\}}$ = line bun on X corr. to ξ.

$X \cong (X^t)^t \leftarrow$

$\mathcal{P}_{X^t}$ on $X^t \times X$ is just $(sw)^* \mathcal{P}$

$sw : X^t \times X \longrightarrow X \times X^t$

$\wp := c_1(\mathcal{P}) \in CH^1(X \times X^t)$

$\mathrm{ch}(\mathcal{P}) := \exp(\wp) = 1 + \wp + \frac{1}{2}\wp^2 + \cdots$

$\cdots + \frac{1}{(2g)!}\wp^{2g}$

$\cdot$-prod

$\in CH(X \times X^t)_{\mathbb{Q}}$

<u>Def</u> Fourier transform

$$\mathcal{F} = \mathcal{F}_X : CH(X)_{\mathbb{Q}} \longrightarrow CH(X^t)_{\mathbb{Q}}$$

$$\begin{tikzcd} & X \times X^t \arrow[dl, "pr_X"'] \arrow[dr, "pr_{X^t}"] & \\ X & & X^t \end{tikzcd}$$

$$\mathcal{F}(\alpha) = pr_{X^t,*}\left(pr_X^*(\alpha)\cdot ch(P)\right)$$

$$\mathcal{F}^t = \mathcal{F}_{X^t} : CH(X^t)_{\mathbb{Q}} \longrightarrow CH(X)_{\mathbb{Q}}$$

<u>THEOREM</u>: (Mukai, Beauville)

(i) $\mathcal{F}^t \circ \mathcal{F} = (-1)^g \cdot [-1]_*$

($n \in \mathbb{Z}$, $[n]_X : X \longrightarrow X$ mult by n ma

Hence $\mathcal{F} : (CH(X)_{\mathbb{Q}}, *) \xrightarrow{\sim} (CH(X^t)_{\mathbb{Q}}, \cdot$

(ii) $\mathcal{F}(\alpha * \beta) = \mathcal{F}(\alpha)\cdot\mathcal{F}(\beta)$

$\mathcal{F}(\alpha\cdot\beta) = (-1)^g \cdot \mathcal{F}(\alpha) * \mathcal{F}(\beta)$