2. Stratifications of $\mathcal{A}_g$

$$k = \overline{\mathbb{F}}_p \supseteq \mathbb{F}_q \supseteq \mathbb{F}_p, \quad K$$

Recap

$$\begin{tikzcd}
\begin{array}{c}\text{p-rank } f(X) \\ (|X[p]| = p^f)\end{array} & & \begin{array}{c}\text{a-number } a(X) \\ (\dim_k \mathrm{Hom}(\alpha_p, X))\end{array} \\
\underbrace{\begin{array}{c}\text{Newton polygon} \\ (X[p^\infty]/\sim \;\Rightarrow\; \text{slopes})\end{array}}_{\text{isogeny invariants}} \arrow[u, Rightarrow, "\#\text{ zero slopes}"'] & & \underbrace{\begin{array}{c}\text{EO-type} \\ (X[p]/\cong \;\Rightarrow\; \varphi)\end{array}}_{\text{isomorphism invariants}} \arrow[u, Rightarrow, "g-\varphi(g)"'] \arrow[ull, Rightarrow, "\max\{i : \varphi(i)=i\}"]
\end{tikzcd}$$

Today, we'll study how these invariants vary in families.

<u>Def 2.3</u> Let $\mathcal{A}_g$ $(= \mathcal{A}_{g,1,1} \otimes \mathbb{F}_p)$ be the moduli space of g-dim principally polarised AV's in char p.

<u>Def 2.11</u> We'll see how each invariant gives a <u>stratification</u> of $\mathcal{A}_g$, i.e. a partition into finitely many locally closed subsets.

Facts about $\mathcal{A}_g$:

<u>Cor 2.5</u>

1) It is a coarse moduli space.

In particular,

$$\mathcal{A}_g(k) \overset{1:1}{\longleftrightarrow} \{(X,\lambda) \text{ } g\text{-dim ppAV}\}/\simeq_k$$

<u>Thm 2.6</u>

2) $\mathcal{A}_g$ is quasi-projective, irreducible, dim $\frac{g(g+1)}{2}$.

Example For elliptic curve over K, the j-invariant determines its $\bar{k}$-isomorphism class.

So $\mathbb{A}^1_{\bar{k},j}$ is the moduli space of elliptic curves over $\bar{\mathbb{F}}_p$.

Over $\mathbb{C}$ may also consider

$$\Gamma \backslash \mathbb{H} = SL_2(\mathbb{Z}) \backslash \{z \in \mathbb{C} : \mathrm{im}(z) > 0\}$$

A. p-rank stratification

Def 2.13 For $0 \leq f \leq g$, let

$$V_f = \{ (X, \lambda) \in \mathcal{A}_g(k) : f(X) \leq f \}$$

be the closed <u>p-rank (f) stratum</u>,

$$V_f^{\circ} = \{ \text{—''—} = f \}$$

Thm 2.15/2.18 (Koblitz-Norman-Oort)

Let $W_f \subseteq V_f$ be an irreducible component.

- $\operatorname{codim}(W_f) = g - f$ in $\mathcal{A}_g$
- generic point (non-ordinary) has a-number 1.

Example $(g=3)$

A_3 has dim $\frac{3(3+1)}{2} = 6$

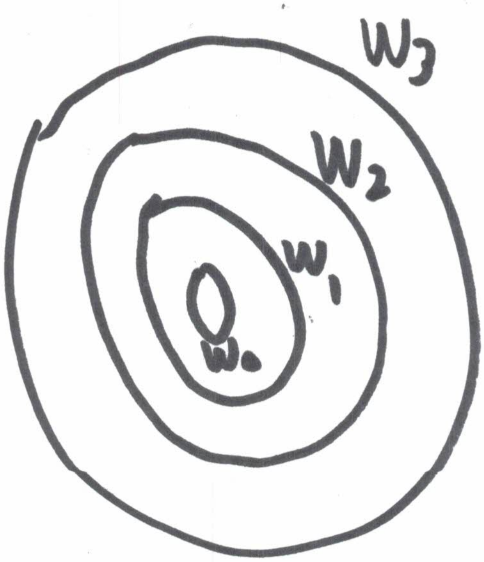
$V_3^0 = \{\text{ordinary AVs}\}$ dim 6

$V_2^0 = \{\text{almost-ord AVs}\}$ 5

$V_1^0 = \{\text{p-rank 1 AVs}\}$ 4

$V_0 = \{\text{p-rank 0 AVs}\}$ 3

we'll see that ss AV's have dim 2!



B. Newton (polygon) stratification

AV X has symmetric Newton polygon $\mathcal{N}(X)$

Def 2.20 For any symmetric Newton polygon ξ, let

$$W_\xi = \{ (X,\lambda) \in \mathcal{A}_g(k) : \mathcal{N}(X) \prec \xi \}$$

be the closed (ξ-) Newton stratum.

$$W_\xi^0 = \{ \quad -\text{''} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_} \quad = \xi \}$$

These are always non-empty.

Thm 2.34/2.35 (Chai-Oort)

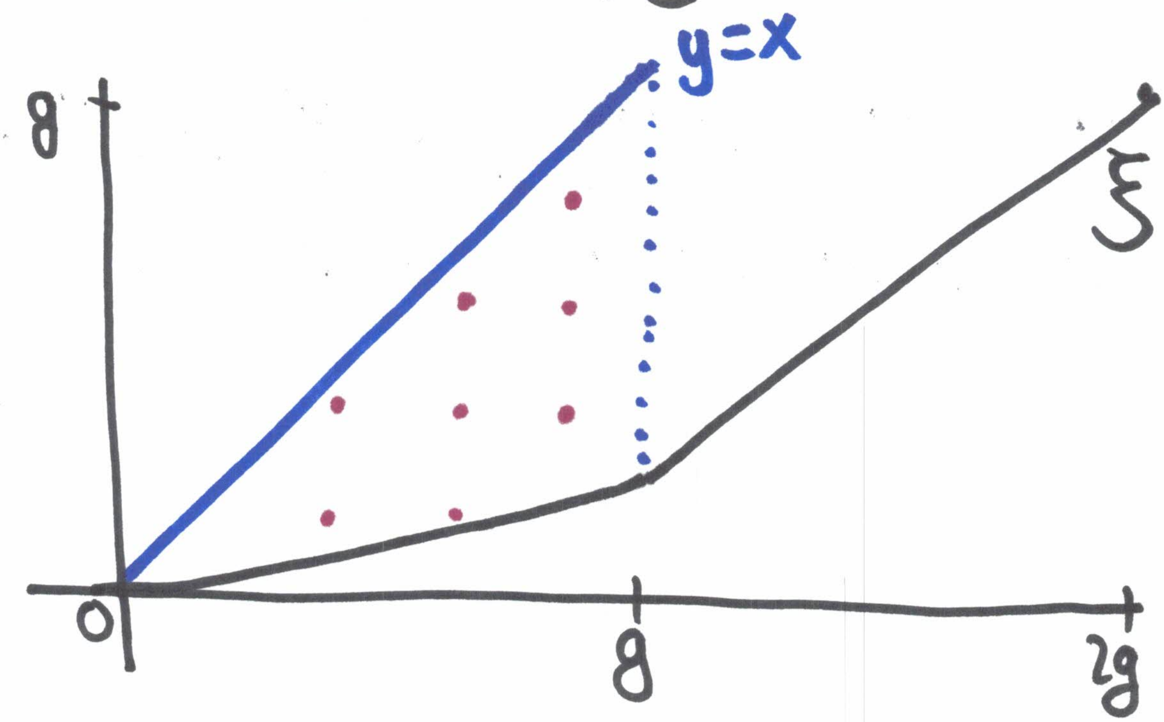
- When $\xi \neq \sigma$, W_ξ is irreducible.
- Let $W \subseteq W_\xi$ be an irreducible component; its generic NP is ξ.
- When $\xi \neq \rho$, generic a-number is 1.

(when $\xi = \rho$, generic a-number is 0.)

Thm (ctd)

: $\dim(W_\xi) = |\Delta(\xi)|$, where

$$\Delta(\xi) = \left\{ \begin{array}{l} (x,y) \in \mathbb{Z}\times\mathbb{Z} : y \leq x \leq g, \\ (x,y) \prec \xi \end{array} \right\}$$



Thm (Grothendieck-Oort)

NP goes up $\Longleftarrow$ specialisation

Example $(g=3)$

Possible Newton polygons

$$\begin{array}{ll}
\sigma = \left(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}\right) & \dim 2 \\[1ex]
\xi_1 = \left(\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3},\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3}\right) & 3 \\[1ex]
\xi_2 = \left(0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1\right) & 4 \\[1ex]
\xi_3 = \left(0,0,\tfrac{1}{2},\tfrac{1}{2},1,1\right) & 5 \\[1ex]
\rho = \left(0,0,0,1,1,1\right) & 6
\end{array}$$

V_0 V_1 V_2 V_3

C. a-number stratification

Def 2.36 For $0 \leq n \leq g$, let

$$T_n = \{ (X,\lambda) \in A_g(\bar{k}) : a(X) \geq n \}$$

be the locally closed a-number n stratum

$$T_n^0 = \{ \text{—''—} = n \}$$

Theorem (Ekedahl - v/d Geer)

For $n \leq g-1$, T_n is irreducible.

(For $n=g$, $T_g = \{\text{superspecial AVs}\}$, dim 0.)

Example $(g=3)$

$T_3 = T_3^0 = \{\text{superspecial AVs}\}$ dim 0

$T_2 = T_2^0 \sqcup T_3^0$

$T_1 = T_1^0 \sqcup T_2^0 \sqcup T_3^0$

$T_0 = \mathcal{A}_3$ dim 6

D. EO-stratification

Def 2.50 For elementary sequence φ,
let $\mathcal{S}_\varphi = \{ (X,\lambda) \in \mathcal{A}_g(k) : \text{the elementary sequence for } X[p] \text{ is } \varphi \}$
be the locally closed Ekedahl-Oort (EO) stratum for φ.

Thm 2.51 (Ekedahl-Oort-v/d Geer-Harashita)

- S_φ is non-empty for all φ and quasi-affine.
- Every irreducible component of S_φ has dimension $\sum_{i=1}^{g} \varphi(i)$.
- S_φ is irreducible $\Longleftrightarrow$ S_φ contains non-supersingular AV's.

Example ($g=3$)

Elt seq.	dim	f	a	irr?
$(0,0,0)$	0	0	3	✗
$(0,0,1)$	1	0	2	✗
$(0,1,1)$	2	0	2	✓
$(0,1,2)$	3	0	1	✓
$(1,1,1)$	3	1	2	✓
$(1,1,2)$	4	1	1	✓
$(1,2,2)$	5	2	1	✓
$(1,2,3)$	6	3	0	✓

So T_3 dim 0, $T_2 = T_2^\circ \sqcup T_3^\circ$ dim 3, $T_1 = T_1^\circ \sqcup T_2^\circ \sqcup T_3^\circ = 5$, T_0 dim 6.

$$
\left.
\begin{array}{lll}
W_\sigma & \supseteq & \left\{ \begin{array}{l} (0,0,0) \\ (0,0,1) \end{array} \right. \\
W_{\xi_1} & \supseteq & (0,1,1) \\
W_\sigma \cup W_{\xi_1} & \supseteq & (0,1,2)
\end{array}
\right\} V_0 \quad \text{dim } 3
$$

$$
W_{\xi_2} \supseteq \left\{ \begin{array}{l} (1,1,1) \\ (1,1,2) \end{array} \right\} V_1^0 \quad \text{dim } 4
$$

$$
W_{\xi_3} \supseteq (1,2,2) \; \} \; V_2^0 \quad \text{dim } 5
$$

$$
W_\rho \supseteq (1,2,3) \; - \; V_3^0 = T_0^0 \quad \text{dim } 6
$$

NP

p-rank