MODEL THEORY PROBLEM SET 2

Beginner problems

Question 1: Let $\mathcal{L}_r = \{0, 1, +, \times, -\}$ denote the language of rings. Is $\mathbb{R}$ an elementary substructure of $\mathbb{C}$ in this language?

Question 2: Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Show that
(a) If $A, B \subseteq M^n$ are definable sets, then so are $A \cup B$, $A \cap B$ and $M^n \backslash A$.
(b) If $A \subseteq M^n$ and $B \subseteq M^m$ are definable sets, then so is $A \times B$.
(c) If n and m are positive integers satisfying $m < n$, $A \subseteq M^n$ and $B \subseteq M^m$ are definable sets, and $\mathrm{pr} : M^n \to M^m$ is the coordinate projection onto the first m coordinates, then $\mathrm{pr}(A)$ and $\mathrm{pr}^{-1}(B)$ are also definable.

Question 3: Let $\mathcal{L} = \{0, 1, +, \times, <\}$, and consider $\mathbb{N}$ as an $\mathcal{L}$-structure. Show that any definable subset of $A \subseteq \mathbb{N}^m$ is can be defined by an $\mathcal{L}$-formula which doesn't use any parameters.

Question 4: Let $\mathcal{K} \subseteq \mathcal{M} \subseteq \mathcal{N}$ be $\mathcal{L}$-structures.
(a) Show that if $\mathcal{K} \preceq \mathcal{M}$ and $\mathcal{M} \preceq \mathcal{N}$, then $\mathcal{K} \preceq \mathcal{N}$.
(b) Show that if $\mathcal{K} \preceq \mathcal{N}$ and $\mathcal{M} \preceq \mathcal{N}$, then $\mathcal{K} \preceq \mathcal{M}$.

Question 5: Suppose that for each $i \in \mathbb{N}$, $\mathcal{M}_i$ is an $\mathcal{L}$-structure, and that $\mathcal{M}_i \preceq \mathcal{M}_{i+1}$ (such a sequence is called an *elementary chain*). There is a natural $\mathcal{L}$-structure $\mathcal{M}$ on $M := \bigcup_{i \in \mathbb{N}} M_i$ (if the structure is not clear, take some time to work out what it should be). Prove that
(a) for every $i \in \mathbb{N}$, $\mathcal{M}_i \preceq \mathcal{M}$, and
(b) if $\mathcal{N} \succeq \mathcal{M}_i$ for every $i \in \mathbb{N}$, then $\mathcal{M} \preceq \mathcal{N}$.

Intermediate problems

Question 6: Consider the language $\mathcal{L} := \{0, 1, +, \times, -, <, f\}$, where f denotes a function symbol in one variable. We can think of $\mathbb{R}$ as an $\mathcal{L}$-structure by choosing a function $F : \mathbb{R} \to \mathbb{R}$, where f is interpreted as F and the other symbols have their usual interpretations. Show that the set of points where F is discontinuous is definable.

Question 7: Let $\mathcal{L} = \{0, +\}$, and consider $\mathbb{Z}$ as an $\mathcal{L}$-structure. Let m be a positive integer. Show that the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = mx$ is not an elementary embedding.

Question 8: Let $\mathcal{L} = \{0, 1, +, \times, -, <\}$, and consider $\mathbb{R}$ as an $\mathcal{L}$-structure. Let $A \subseteq \mathbb{R}^n$ be a definable set. Show that the closure of A (in the Euclidean topology) is also a definable set.

Question 9: Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and let $A \subseteq \mathcal{M}$. Given an element $a \in M$, we a is A-definable to mean that $\{a\}$ is A-definable. We can then define the *definable closure of A in $\mathcal{M}$* to be $dcl^{\mathcal{M}}(A) := \{a \in M : a \text{ is } A\text{-definable}\}$.
(a) Suppose that $\mathcal{M} \preceq \mathcal{N}$ and $A \subseteq M$. Prove that $dcl^{\mathcal{M}}(A) = dcl^{\mathcal{N}}(A)$.
(b) Show that the above can fail if we only assume that $M \subseteq N$ and $\mathcal{M} \equiv \mathcal{N}$.

Advanced problems

Question 10: A structure $\mathcal{M}$ is called *ultrahomogeneous* if any isomorphism between finitely generated substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$. Prove that $(\mathbb{Q}, <)$ is homogeneous by showing that for any two finite substructures $A, B \subseteq Q$ and any isomorphism $\rho : A \to B$, there is an automorphism $\tilde{\rho} : \mathbb{Q} \to \mathbb{Q}$ extending ρ.

Question 11: A structure $\mathcal{M}$ is called *rigid* if the only automorphism of $\mathcal{M}$ is the identity map. Prove that the field $(\mathbb{R}, +, -, \times, 0, 1)$ is rigid. Hint, first, try showing that the field $(\mathbb{Q}, +, -, \times, 0, 1)$ is rigid. Then show that the *ordered* field $(\mathbb{R}, +, -, \times, 0, 1, <)$ is rigid. Finally, use that the ordering on $\mathbb{R}$ is definable in the field $(\mathbb{R}, +, -, \times, 0, 1)$.