INTRODUCTION TO MODEL THEORY WITH APPLICATIONS

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2. Theories, models and definable sets

Modern model theory is often describe as "the study of definable sets in a given structure or a given model of some theory". In the second lecture, we give all the necessary background to make sense of this description. Namely we will look more closely at the notion of a *definable set* and give the definition of a first order theory and its models. All along, we will provide several examples. Finally we will explore some of the natural questions that arise when working with these notions.

2.1. **Definable sets continued.** We ended the first lecture by giving the definition of a definable set. We now give the more general definition of definable sets where one also allows for "parameters". Throughout, $\mathcal{L}$ will denote a fixed language.

Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and let $A \subseteq M$. We would like to think of A (which could be the set M itself) as a set of parameters. We do so by considering the new language $\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$, where each c_a is a new constant symbol. Then, for any $\mathcal{L}$-structure $\mathcal{N} \supseteq \mathcal{M}$ we obtain an L_A-structure, denoted $\mathcal{N}_A$, using the interpretation $c_a^{\mathcal{N}} = a$.

Important.

(1) If $\phi(\overline{x})$ is an $\mathcal{L}_A$ formula, then we will sometime write $\phi(\overline{x}, c_{a_1}, \ldots, c_{a_n})$, where $\phi(\overline{x}, \overline{y})$ is an $\mathcal{L}$-formula in free variables $(\overline{x}, \overline{y})$, to highlight that $c_{a_1}, \ldots, c_{a_n}$ are the constant symbols appearing in $\phi(\overline{x})$.
(2) Moreover, we will abuse notation and think of $a \in A$ itself as the constant symbol c_a. So we will write formulas as $\phi(\overline{x}, a)$ instead of $\phi(\overline{x}, c_a)$.

Definition 2.1. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. A set $Y \subseteq M^n$ is said to be **definable over** A or A**-definable** if it is of the form

$$Y = \{\overline{b} \in M^n : \mathcal{M}_A \models \phi(\overline{b})\}$$

where $\phi(\overline{x})$ is an $\mathcal{L}_A$-formula with free variables $\overline{x} = (x_1, \ldots, x_n)$. Equivalently, $Y \subseteq M^n$ is definable over A if there is an $\mathcal{L}$-formula $\phi(\overline{x}, \overline{y})$ in free variable $(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and $\overline{a} \in A^m$ such that

$$Y = \{\overline{b} \in M^n : \mathcal{M} \models \phi(\overline{b}, \overline{a})\}$$

In any case we will write $Y = \phi(\mathcal{M})$ and say that ϕ defines Y in $\mathcal{M}$.

Let us look at few examples

Example 2.2. Let us work with the language $\mathcal{L} = \mathcal{L}_g$ and let $\mathcal{G} = (G, *, e)$ be a group. Let $h \in G$ be any element and consider the $\mathcal{L}_{\{h\}}$ formula $\phi(x, h) := (x * h = h * x)$. Then we have that

$$\phi(\mathcal{G}) = \{g \in G : \mathcal{G} \models \phi(g, h)\} = \{g \in G : g * h = h * g\}$$

is a definable set and is the centralizer $C_{\mathcal{G}}(h)$ of h in G.

Example 2.3. Consider the field $(\mathbb{C}, +, -, \times, 0, 1)$ and let $P_1, \dots, P_k \in \mathbb{C}[x_1, \dots, x_n]$ be polynomials. Then the set V defined by

$$V = \{\bar{a} \in \mathbb{C}^n : P_1(\bar{a}) = \cdots = P_k(\bar{a}) = 0\}$$

is a so called *algebraic set* and is of course (quantifier-free) definable over $\mathbb{C}$. As is well-known, these set are the closed sets in the Zariski topology on $\mathbb{C}^n$.

Example 2.4. Consider the ordered field $(\mathbb{R}, +, -, \times, 0, 1, <)$ and let $P_1, \dots, P_k \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials. Then the set V defined by

$$V = \{\bar{a} \in \mathbb{R}^n : P_i(\bar{a}) \geq 0 \text{ for all } i : 1, \dots, k\}$$

is a so called *semi-algebraic set*[1] and is of course (quantifier-free) definable over $\mathbb{R}$.

Here are some more concrete examples.

Example 2.5. Consider the field $\mathscr{R} = (\mathbb{R}, +, -, \times, 0, 1)$. It turns out that even though it is not present in the language, we can define the ordering $<$ in this structure. Indeed, consider the formula

$$\phi(x, y) := \exists z((z \neq 0) \wedge y = x + z^2).$$

Then it is not hard to see that for $(a, b) \in \mathbb{R}$, we have that $a < b$ if and only if $\mathscr{R} \models \phi(a, b)$. So $\phi(\mathscr{R})$ is the ordering on the real numbers.

Example 2.6. Recall that Lagrange's theorem says that an integer is positive if and only if it is the sum of four squares. Hence like in the previous example, the ordering is defined in the ring $(\mathbb{Z}, +, -, \times, 0, 1)$ but this time by:

$$\exists z_1 \exists z_2 \exists z_3 \exists z_4((z_1 \neq 0) \wedge y = x + z_1^2 + z_2^2 + z_3^2 + z_4^2).$$

It can be much harder to show that a set is not definable in some given structure. We will come back to this issue soon.

2.2. **Theories and Models.** We can finally bring in the axioms. Recall first that an $\mathcal{L}$-sentence ϕ is an $\mathcal{L}$-formula that has no free variables. Given an $\mathcal{L}$-structure $\mathcal{M}$, if when we go through Definition 1.18 (using the convention $M^0 = \{\emptyset\}$ and $\phi = \phi(\emptyset)$) we get that $\mathcal{M} \models \phi$, then we say that ϕ is true in $\mathcal{M}$. Otherwise we say that ϕ is false in $\mathcal{M}$.

Definition 2.7. By an $\mathcal{L}$**-theory** we simply mean a set of $\mathcal{L}$-sentences. Given such an $\mathcal{L}$-theory T, we say that an $\mathcal{L}$-structure $\mathcal{M}$ is a **model of** T and write $\mathcal{M} \models T$, if $\mathcal{M} \models \phi$ for all $\phi \in T$.

[1]We use $a \geq b$ as an abbreviation for $(b < a) \vee (a = b)$

Notice that a theory T might not have any model. Take for example $T = \{(\forall x\, x = 0), (\exists x\; x \neq 0)\}$ which clearly cannot have any model. We say that a theory T is *consistent* if it has a model. Also we say that a collection $\mathcal{K}$ of $\mathcal{L}$-structures is *axiomatisable* if there is an $\mathcal{L}$-theory T such that $\mathcal{K} = \{\mathcal{M} : \mathcal{M} \models T\}$.

Let us go back to Example 1.6 (where we look at various examples of languages) and give the relevant axiomatization.

Example 2.8.

(1) Consider $\mathcal{L}_\emptyset = \emptyset$ the language of pure sets. Clearly, a set (or more precisely an $\mathcal{L}_\emptyset$-structure) X is infinite if and only if every sentence in $T_\infty = \{\phi_n : n \in \mathbb{N}_{>0}\}$, where

$$\phi_n := \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j,$$

is true in X. Hence models of T_∞ are precisely the infinite sets and so by definition we say that the theory T_∞ *axiomatizes* the theory of infinite sets.

(2) Let $\mathcal{L}_g = \{*, e\}$ be the language of groups. We let T_g be the $\mathcal{L}_g$-theory given by

$$\forall x\, (e * x = x * e = x)$$
$$\forall x \forall y \forall z\, ((x * y) * z = x * (y * z))$$
$$\forall x \exists y\, (x * y = y * x = e).$$

It follows that an $\mathcal{L}_g$-struture $\mathcal{M}$ is a group if and only if $\mathcal{M} \models T_g$. So the theory of groups is axiomatized by T_g.

(3) Let $\mathcal{L}_r = \{+, -, \times, 0, 1\}$ be the language of rings and $\mathcal{L}_{or} = \mathcal{L}_r \cup \{<\}$ be that of ordered rings. Repeating the above observations, we see that the theories of rings and/or fields, as well as their ordered counterparts are all axiomatized by the usual/relevant $\mathcal{L}$-sentences.

Going back to the example of groups (i.e. models of T_g), by experience we know that not all groups are of the same kind. Take for example A an Abelian group and N a non-Abelian one. We have by definition that $A, N \models T_g$ but that $A \models \forall x \forall y\, (x * y = y * x)$ while $N \models \neg(\forall x \forall y\, (x * y = y * x))$. In other words A and N have different L_g-theories in the following sense: given an $\mathcal{L}$-structure $\mathcal{M}$, the set of all $\mathcal{L}$-sentences true in $\mathcal{M}$ is called the **theory of** $\mathcal{M}$ and is denoted by $Th(\mathcal{M})$.

One of our aims is to try and understand $Th(\mathcal{M})$. We are naturally brought to the following concepts

Definition 2.9. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are said to be **elementarily equivalent**, written $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$.

By the above discussion we see that models of the theory of groups are not necessarily elementary equivalent. Naturally, we would like to characterize those theories for which all their models are elementarily equivalent. Given an $\mathcal{L}$-theory T and an $\mathcal{L}$-sentence ϕ, we say that ϕ is *a consequence of* T, and write $T \models \phi$, if for every model $\mathcal{M} \models T$, we have that $\mathcal{M} \models \phi$. A theory T is **complete** if for every $\mathcal{L}$-sentence ϕ, either $T \models \phi$ or $T \models \neg\phi$. Here's our first theorem.

Theorem 2.10.

(1) *For any $\mathcal{L}$-structure $\mathcal{M}$, we have that $Th(\mathcal{M})$ is a complete consistent theory which contains all its consequences.*
(2) *For any consistent theory T. The following are equivalent:*
  i) *T is complete*
  ii) *The set of consequences of T is of the form $Th(\mathcal{M})$ for any $\mathcal{M} \models T$*
  iii) *Any two models of T are elementarily equivalent.*

Proof. 1) For any $\mathcal{L}$-sentence ϕ it follows that $\mathcal{M} \models \phi$ or $\mathcal{M} \models \neg\phi$. Hence $Th(\mathcal{M})$ is complete. The rest follows by definition.
2) i)⇒ii) Assume T is complete and let T' be its set of consequences. Let $\mathcal{M} \models T$. By the definition of $Th(\mathcal{M})$ we must have that $T' \subseteq Th(\mathcal{M})$. We show that $Th(\mathcal{M}) \subseteq T'$. So let $\phi \in Th(\mathcal{M})$. Since T is complete we have that $T \models \phi$ or $T \models \neg\phi$. In the first case, we get that $\phi \in T'$ and are done. We show that the second case cannot happen. Indeed if $T \models \neg\phi$, then it must be by definition that $\mathcal{M} \models \neg\phi$ so we get, since $\phi \in Th(\mathcal{M})$, that $\mathcal{M} \models \phi \wedge \neg\phi$ a contradicton.
ii)⇒iii) Let $\mathcal{M}$ and $\mathcal{N}$ be models of T and let T' be set of consequences of T. The assumption ii) tells us that $Th(\mathcal{M}) = T' = Th(\mathcal{N})$. Hence, $\mathcal{M} \equiv \mathcal{N}$.
iii)⇒i) Assume that for any models $\mathcal{M}$ and $\mathcal{N}$ of T we have that $\mathcal{M} \equiv \mathcal{N}$. Let ϕ be an $\mathcal{L}$-sentence. We need to show that $T \models \phi$ or $T \models \neg\phi$. If $T \not\models \phi$ then by definition there is $\mathcal{M} \models T$ such that $\mathcal{M} \models \neg\phi$. So $\neg\phi \in Th(\mathcal{M})$. For any other model $\mathcal{N} \models T$ we have that $\neg\phi \in Th(\mathcal{N}) = Th(\mathcal{M})$ and hence $\mathcal{N} \models \neg\phi$. So we have shown that $T \models \neg\phi$, i.e. T is complete. □

Remark 2.11. Notice that without the assumption "T is complete", the assertion $T \not\models \phi$ does not necessarily imply that $T \models \neg\phi$. Indeed in the case of groups above, if $\phi := \forall x \forall y\, (x * y = y * x)$ then $T_g \not\models \phi$ and $T_g \not\models \neg\phi$. This is true on the other hand for $\mathcal{M} \not\models \phi$. I have seen this confusion numerous times.

So Theorem 2.10 tells us that complete theories are precisely those theories all models of which are elementarily equivalent. We will later comeback to methods that will help us determine whether a given theory is complete.

2.3. **Elementary substructures and embeddings.** Consider now the special case where we are given two $\mathcal{L}$-structures, $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M} \subseteq \mathcal{N}$. So far we have asked whether $Th(\mathcal{N}) = Th(\mathcal{M})$ [2]. However, since $M \subseteq N$, we can also look at the language $\mathcal{L}_M$ and ask whether $Th(\mathcal{N}_M) = Th(\mathcal{M}_M)$, that is whether the $\mathcal{L}_M$-structures $\mathcal{N}_M$ and $\mathcal{M}_M$ are elementarily equivalent (we write $\mathcal{M} \equiv_M \mathcal{N}$). In other words, we have come up with a notion of elementary substructure.

Important. In what follows, we will use the fact that an $\mathcal{L}_M$-sentence ϕ can also be written in the form $\psi(\overline{a})$ where ψ is an $\mathcal{L}$-formula and $\overline{a} \in M^n$.

[2] For example we will come back to the non-trivial fact that indeed $Th(\mathbb{Q}^{alg}, +, -, \times, 0, 1) = Th(\mathbb{C}, +, -, \times, 0, 1)$.

Example 2.12. In $\mathcal{L}_r = \{+, -, \times, 0, 1\}$, consider the formula $\phi(y) := \exists x(x \times x = y)$. It is not hard to see that $\mathbb{Q} \not\models \phi(2)$. On the other hand $\mathbb{R} \models \phi(2)$. So the two structures are not elementary in the above sense.

It turns out that the quantifiers are the culprit in all examples.

Proposition 2.13. *Suppose $\mathcal{M} \subseteq \mathcal{N}$ are $\mathcal{L}$-structures, $\phi(x_1, ..., x_n)$ is a quantifier-free $\mathcal{L}$-formula, and $\bar{a} \in M^n$. Then $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\bar{a})$.*

Proof. We first argue that if $t = t(\bar{x})$ is an $\mathcal{L}$-term then $t^{\mathcal{M}} = t^{\mathcal{N}}\restriction_{M^n}$. We prove this by induction on the complexity of t. If t is a constant or variable symbol then this is clear. Suppose $t = f(t_1, \ldots, t_{n_f})$, where $f \in L_{\mathcal{F}}$ is a function symbol and $t_1(\bar{x}), \ldots, t_{n_f}(\bar{x})$ are $\mathcal{L}$-terms for which the result is known. Then for any $a \in M^n$,

$$\begin{aligned} t^{\mathcal{M}}(\bar{a}) &= f^{\mathcal{M}}\left(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{M}}(\bar{a})\right) && \text{by definition,} \\ &= f^{\mathcal{M}}\left(t_1^{\mathcal{N}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{N}}(\bar{a})\right) && \text{by induction hypothesis,} \\ &= f^{\mathcal{N}}\left(t_1^{\mathcal{N}}(\bar{a}), \ldots, t_{n_f}^{\mathcal{N}}(\bar{a})\right) && \text{since } f^{\mathcal{M}} = f^{\mathcal{N}}\restriction_{M^{n_f}}, \\ &= t^{\mathcal{N}}(\bar{a}). \end{aligned}$$

Using this we can prove the result using induction on ϕ (which is quantifier-free). If ϕ is of the form $(t_1 = t_2)$ for two $\mathcal{L}$-terms $t_1(\bar{x})$ and $t_2(\bar{x})$, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a}) \\ &\iff t_1^{\mathcal{N}}(\bar{a}) = t_2^{\mathcal{N}}(\bar{a}) \quad \text{since } t_i^{\mathcal{M}} = t_i^{\mathcal{N}}\restriction_{M^n}, \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

If ϕ is of the form $R(t_1, \ldots, t_{n_R})$ for some relation symbol $R \in L_{\mathcal{R}}$ and $t_1(\bar{x}), \ldots, t_{n_R}(\bar{x})$ $\mathcal{L}$-terms, then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff (t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}} \\ &\iff (t_1^{\mathcal{N}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{N}}(\bar{a})) \in R^{\mathcal{M}} \quad \text{since } t_i^{\mathcal{M}} = t_i^{\mathcal{N}}\restriction_{M^n}, \\ &\iff (t_1^{\mathcal{N}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{N}}(\bar{a})) \in R^{\mathcal{N}} \quad \text{since } R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^{k_R}, \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

Next assume that ψ and θ are quantifier-free $\mathcal{L}$-formulas for which the result is known. If $\phi = \neg\psi$ then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \not\models \psi(\bar{a}) \\ &\iff \mathcal{N} \not\models \psi(\bar{a}) \quad \text{by induction,} \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

If $\phi = (\psi \wedge \theta)$ then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \models \psi(\bar{a}) \text{ and } \mathcal{M} \models \theta(\bar{a}) \\ &\iff \mathcal{N} \models \psi(\bar{a}) \text{ and } \mathcal{N} \models \theta(\bar{a}) \quad \text{by induction,} \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

If $\phi = (\psi \vee \theta)$ then

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \models \psi(\bar{a}) \text{ or } \mathcal{M} \models \theta(\bar{a}) \\ &\iff \mathcal{N} \models \psi(\bar{a}) \text{ or } \mathcal{N} \models \theta(\bar{a}) \quad \text{by induction,} \\ &\iff \mathcal{N} \models \phi(\bar{a}). \end{aligned}$$

We are done since ϕ has no quantifiers. □

We our now ready to give the more general definition of "elementary". But first note that rather than only looking at $\mathcal{M} \subseteq \mathcal{N}$ we could made similar inquiries about the situation when there instead is an embedding $\rho : \mathcal{M} \to \mathcal{N}$. Indeed we leave it to the reader to check that Proposition 2.13 (after natural modification) is also true in that case. So

Definition 2.14. Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures. We say that an embedding $\rho : \mathcal{M} \to \mathcal{N}$ is an **elementary embedding** if for all $\mathcal{L}$-formulas $\phi(x_1, ..., x_n)$ and and all $\bar{a} \in M^n$ we have that $\mathcal{M} \models \phi(\bar{a})$ if and only if $\mathcal{N} \models \phi(\rho(\bar{a}))$.
If $\mathcal{M} \subseteq \mathcal{N}$ and the inclusion map is also elementary, we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ and that $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$. We write $\mathcal{M} \preceq \mathcal{N}$.

Theorem 2.15. *Every isomorphism is an elementary embedding.*

Proof. Assume that $\rho : \mathcal{M} \to \mathcal{N}$ is an isomorphism. We leave it as an exercise for the reader to show using induction that the results holds for quantifier free formulas (by modifying the proof of Proposition 2.13). As explained in the first lecture, since $(\neg\exists v \neg)$ abbreviates $(\forall v)$, it suffices to prove the result for $\phi(\bar{x})$ of the form $\exists y \psi(\bar{x}, y)$ and where we assume it already holds for $\psi(\bar{x}, y)$. In this case

$$\begin{aligned} \mathcal{M} \models \phi(\bar{a}) &\iff \mathcal{M} \models \psi(\bar{a}, b) \quad \text{for some } b \in M \\ &\iff \mathcal{N} \models \psi(\rho(\bar{a}), \rho(b)) \quad \text{for some } b \in M \text{ using induction} \\ &\iff \mathcal{N} \models \psi(\rho(\bar{a}), c) \quad \text{for some } c \in N \text{ since } \rho \text{ is surjective} \\ &\iff \mathcal{N} \models \phi(\rho(\bar{a})). \end{aligned}$$

□

Theorem 2.16. *If $\rho : \mathcal{M} \to \mathcal{N}$ is an elementary embedding then $\mathcal{M} \equiv \mathcal{N}$. In particular, isomorphic structures are elementarily equivalent.*

Proof. This follows using the $n = 0$ case of the definition of an elementary embedding. Since isomorphisms are elementary embeddings by the above theorem, isomorphic structures are elementarily equivalent. □

The converse of Theorem 2.16 is false since there are elementarily equivalent substructure that are not elementary substructures. Maybe the simplest example are the structures $(\mathbb{N} \setminus \{0\}, <)$ and $(\mathbb{N}, <)$ which are isomorphic via the map $\rho(n) = n + 1$ (and hence elementarily equivalent by Theorem 2.15). However if we consider the formula $\phi(x) := \exists y(y < x)$, then $(\mathbb{N} \setminus \{0\}, <) \not\models \phi(1)$ while $(\mathbb{N}, <) \models \phi(1)$. We will explore the following concept in more details later:

Definition 2.17. A theory T is said to be **model-complete** if for any models $\mathcal{M}$ and $\mathcal{N}$ of T, if $\mathcal{M} \subseteq \mathcal{N}$ then $\mathcal{M} \preceq \mathcal{N}$.

Finally let us return to the issue of proving that a given set is not definable in some given structure. The following can be useful:

Proposition 2.18. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. If a set $Y \subseteq M^n$ is A-definable and ρ is an $\mathcal{L}$-automorphism of $\mathcal{M}$ fixing A pointwise, then $f(Y) = Y$.*

Proof. Let ρ be an $\mathcal{L}$-automorphism of $\mathcal{M}$ (i.e. $\rho : \mathcal{M} \to \mathcal{M}$ is an $\mathcal{L}$-isomorphism) fixing A pointwise. Suppose Y is defined by $\phi(\bar{x}, \bar{a})$ where $\phi(\bar{x}, \bar{y})$ is an $\mathcal{L}$-formula and $\bar{a} \in A^k$. It follows that

$$\begin{aligned} \bar{b} \in Y &\iff \mathcal{M} \models \phi(\bar{b}, \bar{a}) \\ &\iff \mathcal{M} \models \phi(\rho(\bar{b}), \rho(\bar{a})) \quad \text{by Theorem 2.15} \\ &\iff \mathcal{M} \models \phi(\rho(\bar{b}), \bar{a}) \quad \text{since } \rho(\bar{a}) = \bar{a} \\ &\iff \rho(\bar{b}) \in Y. \end{aligned}$$

□

Example 2.19. We claim that $\mathbb{R}$ is not definable in $(\mathbb{C}, +, -, \times, 0, 1)$. Indeed, if $\mathbb{R}$ was definable, then we will find a finite set A of parameters over which it is defined (the finite complex numbers that appear in the formula). But the transcendence degree of $\mathbb{R}$ over $\mathbb{Q}$, and hence over $\mathbb{Q}(A)$, is infinite. So we can choose $r \in \mathbb{R} \setminus \mathbb{Q}(A)^{alg}$ and $c \in \mathbb{C} \setminus (\mathbb{Q}(A, r)^{alg} \cup \mathbb{R})$ so that in particular r and c are algebraically indpendent over $\mathbb{Q}(A)$. We obtain an isomorphism $\rho : \mathbb{Q}(A, r) \to \mathbb{Q}(A, c)$ which takes r to c and fixes $\mathbb{Q}(A)$. This isomorphism can be extended to an automorphism of $\mathbb{C}$ that fixes $\mathbb{Q}(A)$. But by construction $\rho(\mathbb{R}) \neq \mathbb{R}$ contradicting Proposition 2.18