Hodge Theory

$V_{/\mathbb{C}}$ – alg. var. (sm. proj)

$$H^n(V,\mathbb{Z}) \subset H^n_{betti}(V,\mathbb{C})$$
$$\wr\wr$$
$$H^n_{DR}(V)$$
$$\mathbin{/\!/}$$
$$\bigoplus_{i+j=n} H^{i,j} = \frac{\text{closed } i,j\text{-forms}}{\text{exact } \text{''}}$$

$(i,j): dz_1 \wedge .. \wedge dz_i \wedge d\overline{w_1} \wedge .. \wedge d\overline{w_j}$

Hodge structure wt n

$$= \left\{ \left(L, L_{\mathbb{C}} = \bigoplus_{i+j=n} L^{i,j} \right) \right\}$$

$$\overline{L^{i,j}} = L^{j,i}$$

<u>e.g.</u> E - elliptic curve

$$E = \mathbb{C}/L = \mathbb{C}/\langle 1, \tau \rangle$$

$$H^1(E, \mathbb{Z}) \simeq L^* = \mathrm{Hom}(L, \mathbb{Z})$$

$$H^1_{DR}(E) = \mathbb{C} \cdot dz \oplus \mathbb{C} \cdot \overline{dz}$$

(3)

$$H^1_{DR}(E) \xrightarrow{\sim} H^1(E, \mathbb{C})$$

$$\omega \longrightarrow \left[\gamma \longrightarrow \int_\gamma \omega\right]$$

$$dz \longrightarrow \left[\begin{matrix} [1] \to 1 \\ [\tau] \to \tau \end{matrix}\right]$$

$$H^1(E, \mathbb{C}) / H^{1,0} \simeq \mathbb{C}$$

$$f \longrightarrow f([\tau]) - f([1]) \cdot \tau$$

$$\therefore \frac{H^1(E, \mathbb{C})}{H^{1,0} \oplus H^1(E, \mathbb{Z})} \simeq \frac{\mathbb{C}}{\langle -\tau, 1 \rangle} \simeq E$$

<u>Technical</u>: · Need H^k_{prim} CH^k

· work w/ Polarizations

$Q: L \times L \to \mathbb{Z}$

· Positivity condition

<u>CM Hodge structures</u>

$L^\bullet = (L, L_{\mathbb{C}} = \bigoplus_{i+j=n} L^{i,j})$, $2|n$

$v \in L_{\mathbb{Z}}$ is a <u>hodge class</u> if $v \in L^{n/2, n/2}$

$L^\bullet$ is <u>CM</u> if $(L^\bullet)^{\otimes m}$ has many hodge classes.

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e.g. if E is CM

$$H^1(E,\mathbb{Z})^{\otimes 2} \supset H^2(E\times E, \mathbb{Z})$$
$$\cup$$
$$[\Gamma_\phi],\ \phi\in \mathrm{End}(E)$$

Motivation Given sm. proj. V,

$W\subset V$ $\mathrm{codim}_{\mathbb{C}} W = k$

$[W]\subset H^{2k}(V,\mathbb{Z})$

$\int_W \omega = 0$ if ω is (i,j)

$(i,j)\neq(k,k)$, $\omega|_W = 0$.

<u>Conj</u>: <u>All</u> Hodge classes in $H^{2k}(V)$ are algebraic up to $\mathbb{Q}$-coefficients.

<u>Conj</u> (AO for HS, KLINGLER)
Let $\pi: X \to B$ be sm. proj. family. Fix $k \in \mathbb{N}$. Let

$$B_{CM} = \{ b \in B : \underbrace{H^k(X_b, \mathbb{Z}) \text{ is}}_{CM} \}$$

If $\overline{B_{CM}}^{Zar} = B$, $\exists \phi : B \twoheadrightarrow S$ (Shimura)
s.t. $B_{CM} = \phi^{-1}(S_{CM})$.

Moduli

Fix H.S. $L^{\cdot} = \left(L, L_{\mathbb{C}} = \bigoplus_{i+j=n} L^{i,j}, Q\right)$.

$D = \left\{ \{A^{i,j} \subset L_{\mathbb{C}}\} : (L, L_{\mathbb{C}} = \bigoplus A^{i,j}, Q) \right.$ is a H.S., $\left. \dim A^{i,j} = \dim L^{i,j} \right\}$.

$G = \mathrm{Aut}(L, Q)$, $G(\mathbb{R}) \circlearrowright D$

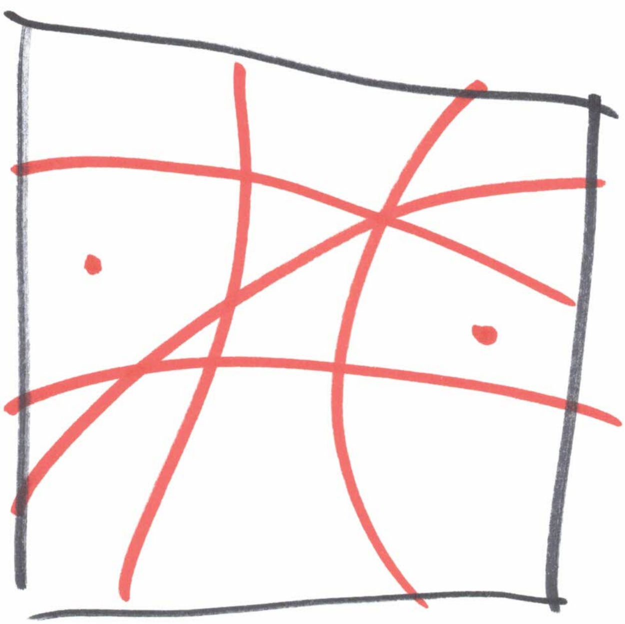
$G(\mathbb{Z}) \backslash D$ – mod. space of H.S.

<u>ISSUE</u>: $G(\mathbb{Z}) \backslash D$ has <u>no</u> alg. structure. (Griffiths + ...)

Given $X \to B$, get

Period maps $\phi : B \to G(\mathbb{Z}) \backslash D$

$$\phi(b) = H^k(X_b).$$



$$\begin{array}{c} G(\mathbb{Z}) \backslash D \\ \cup \\ (G(\mathbb{Z}) \backslash D)^{geom} \end{array}$$

Q: Does $(G(\mathbb{Z}) \backslash D)^{geom}$ have an arithmetic structure?

Galois actions?

<u>Issue:</u> $\exists ?$ $V_{/\mathbb{Q}}, W_{/\mathbb{Q}}$

$H^k(V) \cong H^k(W)$ as H.S.

$\sigma \in \mathrm{Aut}(\mathbb{C})$

s.t. $H^k(\sigma(V)) \not\cong H^k(\sigma(W))$

(<u>Absolute Hodge conj</u>)

$\ell \in H^k_{DR}(V)$ hodge

$\sigma \in \cancel{H^k}$ $\sigma \in \mathrm{Aut}(\mathbb{C})$.

is $\sigma(\ell) \in H^k_{dR}(\sigma(V))$

hodge?

HC $\Longrightarrow$ AHC

$\ell = [W]$, $W \subset V$

$\sigma(\ell) = [\sigma(W)]$, $\sigma(W) \subset \sigma(V)$

Thm (Deligne) AHC true for abelian varieties

(Voisin, Urbanik, ..)

AHC $\Rightarrow$ $\forall\ X \longrightarrow B\ /\ \overline{\mathbb{Q}}$,

B_{CM} defined over $\overline{\mathbb{Q}}$.

Galois action on B_{CM}

would be new reflex maps.

Mixed Hodge structures

$\cup$

Mixed SHIMURA

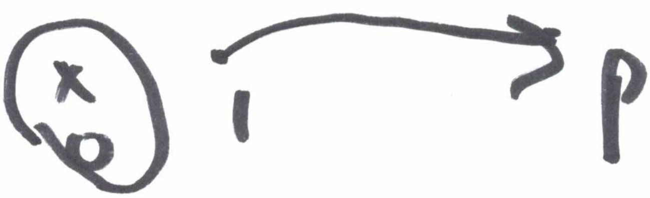
VARIETIES

$H^k(V)$, V not sm. or proper.

<u>example:</u>

$$p \in \mathbb{C}^\times \setminus \{1\}$$

$$H^1_{rel}(\mathbb{C}^\times, \{1, p\})$$



$$\begin{tikzcd}
 & & H^1_{rel}(\mathbb{C}^\times,\{1,p\}) \arrow[d] & & \\
0 \arrow[r] & \mathbb{Z}\cdot([p]-[1]) \arrow[r] & \text{''} \arrow[r] & H^1(\mathbb{C}^\times) \arrow[r] & 0
\end{tikzcd}$$

mod. space is $\mathbb{G}_m$

CM $=$ $\mathbb{G}_m[\mathrm{tor}]$