

Q: $f \in \mathbb{C}[x, y, x', y']$ has ①

∞ -ly many solutions w/ x, y roots of unity. what can f be?

$$(\mathbb{C}^*)^2 \cong \mathbb{C}_m^2(\mathbb{C})$$

$\{(x, y): x, y \text{ roots of unity}\}$

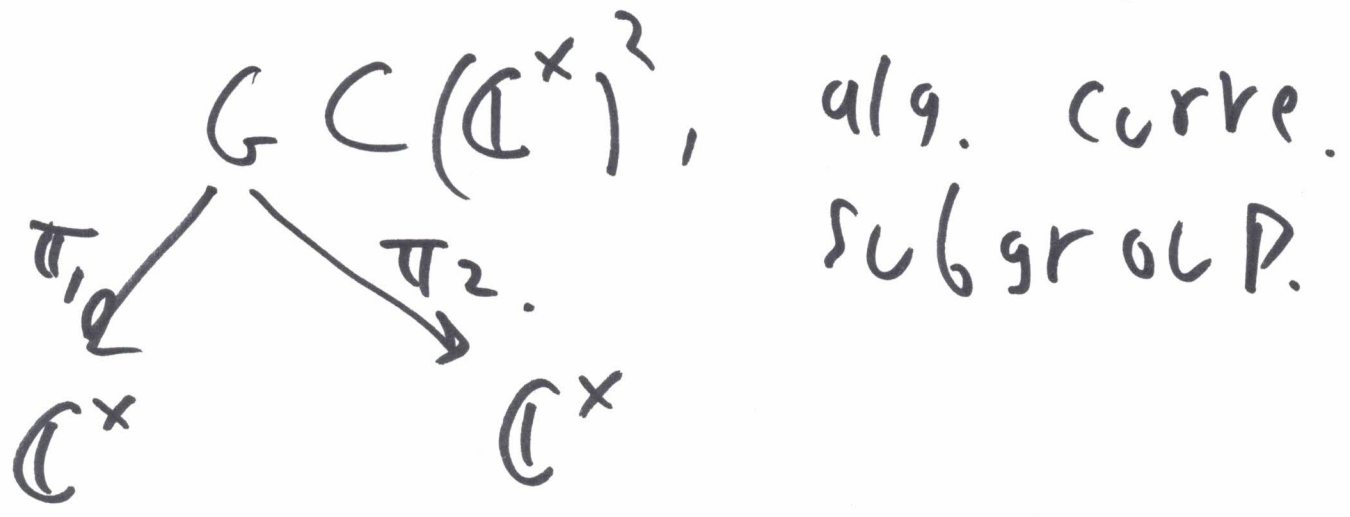
||
 $(\mathbb{C}^*)^2$ tor

e.g. $x^m y^n - \zeta$, $(m, n) \in \mathbb{Z}^2 \setminus \{0\}$
 ζ -r.o.u.

Thm (Lahg) These are all the examples.

(2)

$Z(x^m y^n - 1) \leftarrow$ algebraic subgroup



$$1 \rightarrow K \rightarrow G \xrightarrow{\pi_1} \mathbb{C}^x \rightarrow 1$$

↑ order n .

$$\exists \psi: \mathbb{C}^x \rightarrow G, \quad \pi_1 \circ \psi: \mathbb{Z} \rightarrow \mathbb{Z}^n$$

$$\pi_2 \circ \psi: \mathbb{C}^x \rightarrow \mathbb{C}^x, \quad \mathbb{Z} \rightarrow \mathbb{Z}^m$$

Def! $T \subset G_m^2$ is a $\textcircled{3}$

torus (connected subgroup)

$\xi \in G_m^2 \setminus T$, $\xi \cdot T \leftarrow$ Torsion
Coset.

Thm. $C \subset G_m^2$ irr. curve,

$\#C \cap (C^x)_{\text{tor}} = \emptyset \Rightarrow C$
torsion
Coset.

Galois ORbits

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$$G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$$

$M_n(\mathbb{C}) \subset \mathbb{C}^x$ - n th roots
of unity.

$$M_{\infty}(\mathbb{C}) := \bigcup_n M_n(\mathbb{C})$$

$$\text{Aut}(M_{\infty}(\mathbb{C})) = \varprojlim \text{Aut}(M_n(\mathbb{C}))$$

$$\cong (\mathbb{Z})^x$$

$$\cong (\mathbb{Z}/n\mathbb{Z})^x$$

$G_{\mathbb{Q}}$

Cor: $\vec{\xi} \in (\mathbb{C}^x)^n$, $\text{ord}(\vec{\xi}) = m$ $\textcircled{5}$

$$\#G_{\mathbb{Q}}(\vec{\xi}) = \varphi(m) = m^{1+o(1)}$$

Pf 1 (Intersection)

Let $C \subset G_m^2$ contradicts
the statement.

Assume C/\mathbb{Q}

$$\vec{x} \in C(\mathbb{C}) \cap (\mathbb{C}^x)^2_{\text{tor}}$$

$$\text{ord}(\vec{x}) = m$$

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P -prime, $(P, m) = 1$

$$P = O(\ln m) = m^o(1)$$

$$\left[\begin{array}{l} \text{LTP}'' \sim '' X, \ln \text{TT} P \\ P \leq \ln X, \quad P \leq \ln X \\ \sum_{P \leq \ln X} \ln P'' \sim \ln X \\ \sim \ln \ln X \cdot \frac{\ln X}{\ln \ln X} = \ln X \end{array} \right]$$

(7)

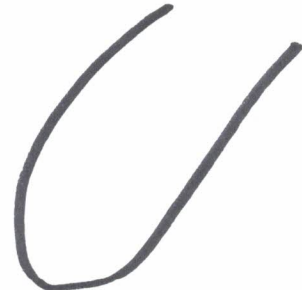
$$\vec{x} \in C \Rightarrow \vec{x}^P \in C$$

$$\Rightarrow \vec{x} \in C \cap C^{1/P}$$

Case 1: $C \neq C^{1/P}$

$$\deg(C^{1/P}) \leq P \cdot \deg(C)$$

Bezout: $\# C \cap C^{1/P} \leq P^2 \deg(C)^2$



$$\Rightarrow d^2 = m^{O(1)}$$

$$\# G_{\mathbb{Q}} \cdot \vec{x} = \phi(m) = m^{1+o(1)}$$

$$\Rightarrow \Leftarrow$$

①

Case 2: $C = C^{1/P}$

Claim: C - torsion closed

wlog $1 \in C$.

$U = \log(C)$ - germ
at 0

$U = PU = P^2U = P^3U = \dots$

$\therefore U$ is linear

$\therefore C$ is a subgroup

\triangleright

Higher dimensions

(9)

Careful: $\Delta \subset \Sigma \subset C(\mathbb{C}^x)$
 \cong
 \mathbb{C}^x

Thm (Laurent, Lang's Cohy)

$V \subset \mathbb{G}_m^n$, irreducible.

$$(V \cap (\mathbb{C}^x)_{\text{tor}}^n)^{\text{zar}} = V$$

$\therefore V$ is a torsion Colet

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Pf sketch ($n=3, \dim V=2$)

Idea: Consider $V \cap V^{(p)}$

Issue: C_1, \dots, C_h, \dots

of torsion (or) curves
in V . $C_i = x_i \cdot T_i$, x_i -tor.

O-minimality look at
"slopes" of the T_i

definable, discrete

\therefore finite.

EQUIDISTRIBUTION

$$m_i = \text{ord}(\bar{x}_i), \bar{x}_i \in (\mathbb{C}^*)^3 / T_i$$

(1) m_i unbounded.

Pick $P,$

$$V \cap V''^P \supset \varphi(m_i) \text{ (set) of } T_i$$

FULTON

$$V_1, \dots, V_m \in \mathbb{P}^n$$

$$\text{deg}(\cap V_i) \leq \prod \text{deg } V_i$$

(2) m_i bounded, Assume

$$m_i = 1.$$

$$T = T^2$$

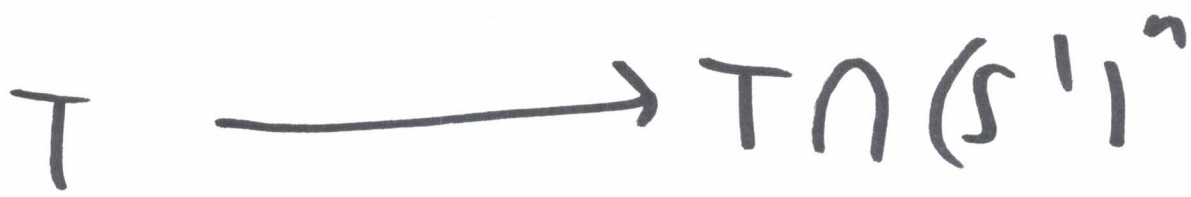
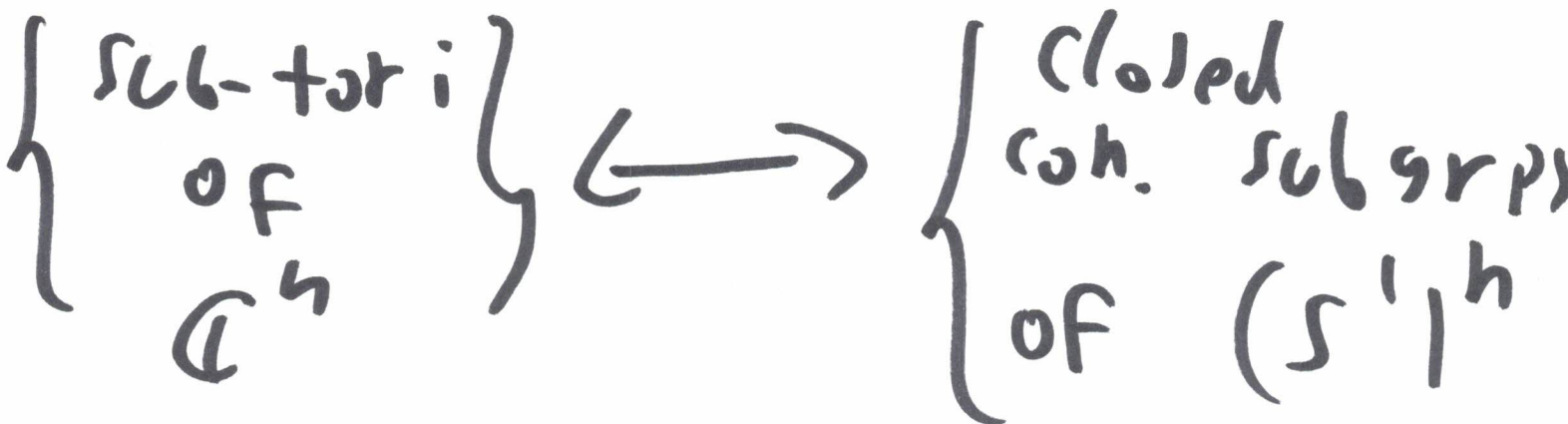
$$\therefore V = V^2$$

$\therefore V$ subgroup.

EGU DISTRIBUTION

$\vec{x} \in (\mathbb{C}^x)^h$ tor, $\delta_{\vec{x}}$ - Delta measure.

$$\mu_{\vec{x}} = \sum_{\vec{z} \in G_{\mathbb{C}} \cdot \vec{x}} \delta_{\vec{z}} = \int_{G_{\mathbb{C}}} \delta_{g\vec{x}} dg$$



$$HC(S^1)^n,$$

μ_H - Haar measure on H

$$\mu_{\vec{x}H} = \int_{G/G} \mu_{g\vec{x}\cdot H} dg$$

Thm (Billu) $\{ \mu_S, \text{S-torsionless} \}$
(over)

is weak \rightarrow closed.

Billu \Rightarrow Lang

PF: $\chi: (S')^h \rightarrow S'$

$\forall k, \int t^k d\mu_n(t) \rightarrow 0$

as $h \rightarrow \infty$.

(ZHANG)

$$f = \sum_{v \in S} a_v x^v, \quad v \in \mathbb{Z}^n$$

$$f(\eta) = 0 \Rightarrow f(\eta^a) = 0$$

Find arithmetic progression
of a 's

(16)

∴ Rändermaße dot.

kaufisches $\Rightarrow \int_{v-u} = 1, v, u \in S.$

torsion  coset.