Q: $f \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ has ∞-ly many solutions w/ x, y roots of unity. What can f be?

①

$$(\mathbb{C}^\times)^2 \cong \mathbb{G}_m^2(\mathbb{C})$$

$$\{(x,y) : x, y \text{ roots of unity}\} = (\mathbb{C}^\times)^2_{tor}$$

<u>e.g.</u> $x^m y^n - \zeta$, $(m,n) \in \mathbb{Z}^2 \setminus 0$, ζ - r.o.u.

<u>Thm</u> (Lang) These are all the examples.

$Z(x^m y^n - 1)$ ← algebraic subgroup

$G \subset (\mathbb{C}^\times)^2$, alg. curve. subgroup.

$$\begin{tikzcd} & G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C}^\times & & \mathbb{C}^\times \end{tikzcd}$$

$$1 \to K \to G \xrightarrow{\pi_1} \mathbb{C}^\times \longrightarrow 1$$

K orden n.

$\exists\, \varphi: \mathbb{C}^\times \longrightarrow G$, $\pi_1 \circ \varphi: z \to z^n$

$\pi_2 \circ \varphi: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$, $z \to z^m$

<u>Def:</u> $T \subset \mathbb{G}_m^2$ is a torus (connected subgroup)

$\xi \in \mathbb{G}_m^2(\mathbb{C})_{tor}$, $\xi \cdot T \leftarrow$ Torsion coset.

<u>Thm.</u> $C \subset \mathbb{G}_m^2$ irr. curve,

$\# C \cap (\mathbb{C}^\times)^2_{tor} = \infty \Leftrightarrow C$ torsion coset.

Galois ORbits

$$G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$$

$\mu_n(\mathbb{C}) \subset \mathbb{C}^\times$ – nth roots of unitS.

$$\mu_\infty(\mathbb{C}) := \bigcup_n \mu_n(\mathbb{C})$$

$$\mathrm{AuT}(\mu_\infty(\mathbb{C})) = \varprojlim \mathrm{AuT}(\mu_n(\mathbb{C}))$$

$$\mathrm{AuT}(\mu_n(\mathbb{C})) \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

$$\mathrm{AuT}(\mu_\infty(\mathbb{C})) = (\hat{\mathbb{Z}})^\times \twoheadleftarrow G_{\mathbb{Q}}$$

5

<u>Cor</u>: $\vec{\zeta} \in (\mathbb{C}^{\times})^n$, $\operatorname{ord}(\vec{\zeta}) = m$

$$\# G_{\mathbb{Q}} \cdot \vec{\zeta} = \varphi(m) = m^{1+o(1)}$$

<u>Pf 1</u> (Intersection)

Let $C \subset \mathbb{G}_m^2$ contradicts the statement.

Assume $C/\mathbb{Q}$

$\vec{x} \in C(\mathbb{C}) \cap (\mathbb{C}^{\times})^2_{\text{tor}}$

$\operatorname{ord}(\vec{x}) = m$

p - prime, $(p, m) = 1$

$p = O(\ln m) = m^{o(1)}$

$\prod_{p \le \ln x} p$ "$\sim$" x, $\ln \prod_{p \le \ln x} p$

"$\sum_{p \le \ln x} \ln p$" $\cancel{\sim \ln x}$

$\gtrsim$

$\ln \ln x \cdot \dfrac{\ln x}{\ln \ln x} = \ln x$

$$\vec{x} \in C \Rightarrow \vec{x}^{p} \in C$$

$$\Rightarrow \vec{x} \in C \cap C^{1/p}$$

Case 1: $C \neq C^{1/p}$

$$\deg(C^{1/p}) \leq p \cdot \deg(C)$$

Bezout: $\# C \cap C^{1/p} \leq p \deg(C)^{2}$

$$= d^{o} = m^{o(1}$$

$$\# G_{Q} \cdot \vec{x} = \phi(m) = m^{1+o(1)}.$$

$$\Rightarrow\Leftarrow$$

<u>Case 2</u>: $C = C^{1/p}$

<u>Claim</u>: C - torsion coset

w/log $1 \in C$.

$U = \log(C)$ - germ at 0

$U = pU = p^2 U = p^3 U = \cdots$

$\therefore U$ is linear

$\therefore C$ is a subgroup

$\triangleright$

Higher dimensions

Careful: $\Delta \subset S \subset (\mathbb{C}^\times)^3$
$$\begin{array}{c}\Delta\\ \wr\wr \\ \mathbb{C}^\times\end{array}$$

Thm (Laurent, Lang's conj)

$V \subset \mathbb{G}_m^n$, irreducible.

$$\left(V \cap (\mathbb{C}^\times)^n_{tor}\right)^{zar} = V$$

$\therefore$ V is a torsion coset

Pf sketch ($n=3$, $\dim V = 2$)

Idea: consider $V \cap V^{||p}$

Issue: $C_1, \ldots, C_n, \ldots$ of torsion coset curves in V. $C_i = x_i \cdot T_i$, x_i - tor.

O-minimality look at "slopes" of the T_i

definable, discrete

$\therefore$ finite.

EQUIDISTRIBUTION

$m_i = \operatorname{ord}(\overline{x_i}),\ \overline{x_i} \in (\mathbb{C}^\times)^3/T_i$

(1) m_i unbounded.

Pick p,

$V \cap V^{1/p} \supset \varphi(m_i)$ (cosets of T_i)

FULTON

$V_1, \ldots, V_m \in \mathbb{P}^n$

$\deg(\cap V_i) \leq \prod \deg V_i$

(2) m_i bounded, Assume

$m_i = 1$.

$T = T^2$

$\therefore$ $V = V^2$

$\therefore$ V subgroup.

EQUIDISTRIBUTION

$\vec{x} \in (\mathbb{C}^\times)^n_{tor}$, $\delta_{\vec{x}}$ - Delta measure.

$$\mu_{\vec{x}} = \sum_{\vec{\zeta} \in G_{\mathbb{Q}} \cdot \vec{x}}' \delta_{\vec{\zeta}} = \int_{G_{\mathbb{Q}}} \delta_{g\vec{x}} \, dg$$

$$\left\{ \begin{matrix} \text{sub-tori} \\ \text{of} \\ \mathbb{C}^n \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Closed} \\ \text{con. subgrps} \\ \text{of } (S^1)^n \end{matrix} \right.$$

$$T \longrightarrow T \cap (S^1)^n$$

$$H^{zar} \longleftarrow H$$

$H \subset (S^1)^n$,

μ_H - haar measure on H

$$\mu_{\vec{x}H} = \int_{G_Q} \mu_{g\vec{x}\cdot H} \, dg$$

Thm (Bilu) $\{\mu_S, \ S$ - torsion coset$\}$ is weak-$*$ closed.

Bilu $\Longrightarrow$ Lang

Pf: $\chi : (S^1)^n \longrightarrow S^1$

$\forall k, \quad \int t^k \, d\mu_n(t) \longrightarrow 0$

as $n \longrightarrow \infty$.

(ZHANG)

$$f = \sum_{v \in S} a_v x^v, \quad v \in \mathbb{Z}^n$$

$$f(\zeta) = 0 \Rightarrow f(\zeta^a) = 0$$

Find arithmetic progression of a's

$\therefore$ vandermonde det.

vanishes $\Rightarrow$ $\zeta^{v-w} = 1$, vikes.

torsion coset.