# MODEL THEORETIC ORIGINS AND APPROACHES TO UNLIKELY INTERSECTION PROBLEMS

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The lectures will be divided into three topics.

Topic 1. How do the conjectures on unlikely intersections relate to the logical theory of the complex exponential function?

Topic 2. What do the conjectures on special points imply about the first-order theory of the complex numbers expanded by sets of special points or special varieties?

Topic 3. How do we use differential algebra to prove theorems on unlikely intersections?

Topic 1 will comprise one lecture. I will explain Boris Zilber's proposed axiomatization of the theory of the complex exponential function and how his Conjecture on the Intersection with Tori is needed to express the axioms. Schanuel's Conjecture and its generalizations to semiabelian varieties and Shimura varieties will appear here.

Topic 2 will be presented in two lectures. This starts with an observation of Anand Pillay that the Mordell-Lang Conjecture may be expressed as a stability theoretic statement about the structure of the complex numbers expanded by a predicate for a finite rank subgroup of a semiabelian variety. Other special point theorems / conjectures have similar interpretations. I will explain how this interpretation can be fed back to give some uniform versions. From this we may deduce implications between different forms of the unlikely intersections conjectures. For example, Zilber-Pink for semiabelian varieties implies Mordell-Lang. I will conclude this topic by explaining how the André-Oort Conjecture is related to the first-order theory of the field of rational functions.

Topic 3 starts with Ax's proof of a functional version of the Schanuel Conjecture. I will then talk about algebraic differential equations to describe special varieties, showing, for example, how to deduce weak Zilber-Pink from Ax-Schanuel. I will close with the Bézout-style bounds coming from differential algebra.

As to projects, in relation to Topic 2, we will work out the first-order theory of the complex numbers expanded by a predicate for the relation "$j$ and $j'$ are related if they are $j$-invariants of isogenous elliptic curves". The necessary geometric input to this project is an effective version of the Zilber-Pink conjecture. The (un)decidability of this structure is related to the first-order theory of the field of rational functions over the complex numbers.

For Topic 3, we will improve the bounds in unlikely intersection theorems beyond those given by the general theory, since as a general rule, the inequalities given by intersection theory are much weaker than what is true. To keep this project manageable, we will restrict most of our attention to low dimensions and explicit cases of special varieties.

For now, I am not proposing a project in connection with Topic 1, though I have asked Boris Zilber for advice and we may add a project before the Winter School strarts.