Error in the "proof" of CIT $\Longrightarrow$ uniform CIT.

Atypical component of generic fibres spread to atypical components of the total space. ✓

Special fibers do <u>not</u> spread. A correct proof appears in J. Pila's "Point counting and the Zilber–Pink conjecture" Thm 24.7 in chapter 24.

<u>Problem</u> Is the theory

of $\mathbb{C}(+|$ decidable?

In $\mathcal{L}(+, \cdot, 0, 1, t)$

can we find an algorithm

to correctly determine given

φ a sentence in $\mathcal{L}(+, \cdot, 0, 1, t)$

whether φ is true in

$\mathbb{C}(+|$?

Related question:

Decidable:

- $\mathbb{R}$
- $\mathbb{C}$
- $\mathbb{Q}_p$
- $\mathbb{C}((t))$

Open:

- $\overline{\mathbb{F}}_p((t))$
- $\mathbb{C}(t)$

Undecidable

- $\mathbb{Q}$
- any # field
- $\overline{\mathbb{F}}_p(t)$
- $\mathbb{Q}_p(t)$
- $\mathbb{C}(V)$ $\dim V > 1$
- $\overline{\mathbb{F}}_p^{alg}(t)$

$\mathbb{C}(t)$ is cohomologically trivial

e.g. every quadratic form represents zero.

Approach suggested by ~~Th~~ Thanasis Pheidas: in $\mathbb{C}(t)$ we can define the set $\{ j(E) : E \text{ has CM} \}$.

CM

CM is a countably infinite set of algebraic integers. This should create complexity.

<u>Prop</u> $\mathbb{C}$ is definable in $\mathbb{C}(+)$

<u>Pf</u>: $\mathbb{C} = \{x \in \mathbb{C}(+) :$

$\exists y \quad x^3 + y^3 = 1 \}$

An elliptic curve E has complex multiplication, CM, if $\operatorname{rk} \mathrm{End}(E) = 2$

$\iff \# \, \mathrm{End}(E) / 2\,\mathrm{End}(E) = 4$

E_1, E_2 two complex elliptic curves

$\mathrm{Mor}(E_1, E_2) = \mathbb{Z}$

ψ

$$\psi(x) = \varphi(x) + p$$

$\varphi : E_1 \to E_2$ is a homomorphism of algebraic group & $p \in E_2(\mathbb{C})$.

The natural maps $E_1 \dashrightarrow E_2$ are all regular.

Rational maps $E_1 \to E_2$

$\longleftrightarrow E_2(\mathbb{C}(E_1))$

$E_j: \quad y^2 = x^3 + A_j x + B_j$

$$\mathbb{C}(E_1) = \mathbb{C}(t)[y] \big/ (y^2 - t^3 - A_1 t - B_1)$$

$$= \mathbb{C}(t) \oplus \mathbb{C}(t) \cdot y$$

$$\cong (\mathbb{C}(t)^2, +, \cdot)$$

coordinatewise

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2)$$

$$= (\alpha_1\alpha_2 + \beta_1\beta_2 \cdot (t^3 + A_1 t + B_1)),$$

$$\alpha_1\beta_2 + \alpha_2\beta_1).$$

We have interpreted $\mathbb{C}(E_1)$ in $\mathbb{C}(t)$.

We can then interpret $E_2(\mathbb{C}(E_1))$ as a set of quadruples of rational function.

Conclusion: The sets of $\mathrm{Mor}(E_1, E_2)$ are uniformly definable in $\mathbb{C}(t)$.

$$0 \to E_2(\mathbb{C}) \to \mathrm{Mor}(E_1, E_2) \to \mathrm{Hom}(E_1, E_2) \to 0$$

$$\mathrm{Mor}(E_1, E_2) = E_2(\mathbb{C}(E_1))$$

$$0 \longrightarrow E_2(\mathbb{C}) \to E_2(\mathbb{C}(E_1)) \to \mathrm{Hom}(E_1, E_2) \to 0$$

$$\mathrm{Hom}(E_1, E_2) \cong E_2(\mathbb{C}(E_1)) / E_2(\mathbb{C})$$

$\mathcal{X}' = \{ (A_1, B_1, A_2, B_2) \in \mathbb{C}^4 :$

$E_j : y^2 = x^3 + A_j x + B_j$ is

an elliptic curve

$4A_j^3 + 27B_j^2 \neq 0$

& $\exists$ non constant

$E_1 \longrightarrow E_2 \}$

is definable in $\mathbb{C}(t)$

Similarly,

$CM' = \{ (A,B) \in \mathbb{C}^2 :$
$E: Y^2 = X^3 + Ax + B$ is
an elliptic curve
w/ $CM \}$

is definable

$\left(\# \frac{E(\mathbb{C}(E))}{2E(\mathbb{C}(E))} = 4 \right)$

$$j^{alg}(E_{A,B}) = \frac{1728\,(4A)^3}{-16\,(4A^3+27B^2)}$$

$$= j(\tau)$$

$$E_{A,B}(\mathbb{C}) = \mathbb{C}/_{\mathbb{Z}+\mathbb{Z}\tau}$$

$$E \cong E' \Longleftrightarrow j^{alg}(E) = j^{alg}(E')$$

$$CM = \{ j(\tau) : \tau \text{ quadratic imaginary} \}$$

$$= \{ j^{alg}(E) : E \text{ has CM} \}$$

CM is definable in

$\mathbb{C}(+)$.

$\mathfrak{M} := (\mathbb{C}, +, \cdot, 0, 1, CM)$

is interpreted in $\mathbb{C}(+)$.

How complicated is it?

<u>Prop</u> $\mathbb{Z}$ are <u>not</u>

definable in $\cancel{\mathbb{C},+}$, ΣM. (Reason: ΣM is ω-stable.)

<u>Pf</u> André-Oort conjecture:

$Th(\Sigma M)$ is determined by the $Th(\mathcal{M}_{ind})$

induced structure on

$\mathbb{C}M$

ie $X \subset \mathbb{C}^n$

definable $\qquad R_X = X \cap \mathbb{C}M^n$

Decidability of $\mathbb{C}M$

would follow from effective

a.o.