Def $\mathbb{C}_{exp} = (\mathbb{C}, +, \cdot, exp, 0, 1)$

$\mathcal{L}(+, \cdot, E, 0, 1)$

$E^{\mathbb{C}_{exp}} = exp$

$exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Def $\mathbb{R}_{exp} = (\mathbb{R}, +, \cdot, exp, 0, 1)$

Thm (Wilkie)

$Th(\mathbb{R}_{exp})$ is model complete & o-minimal

Prop $Th(\mathbb{C}_{exp})$ is undecidable.

Pf $\varphi(x) :=$

$$\forall z [E(z) = 1 \rightarrow E(xz) = 1]$$

$$\varphi(\mathbb{C}_{exp}) = \mathbb{Z},$$

From a decision procedure for $Th(\mathbb{C}_{exp})$

We obtain a decision procedure for $\mathrm{Th}(\mathbb{Z},+,\cdot,0,1)$

Impossible by Gödel. //

We work in

$\mathcal{L}_{\omega_1,\omega}(Q)$

We allow countable

conjunction, countable

disjunction, usual

first operations,

"Qx" — "there are

uncountably many x

s.t. . . . "

$(\exists x_i)_{i<\lambda_0<\lambda}\ \varphi$

$\mathcal{L}_{\kappa,\lambda}$

conjunction of $< \kappa$ formulas

T_{exp} Theory of pseudo-exponentiation

- ELA "exponential, logarithmic, algebraically closed field"

ACF_0 : axioms for algebraically closed fields of characteristic zero.

eg. $\forall x \forall y \forall z$

$x \cdot (y+z) = x \cdot y + x \cdot z$

- $1+1 \neq 0$

  $\neg (1+1 = 0)$

  $1+1+1 \neq 0$

- $\forall a_0 \forall a_1 \ldots \forall a_{n-1}$

  $\exists x \quad x^n + a_{n-1} x^{n-1} + \ldots + a_0 = 0$

$n \in \mathbb{Z}$

- $E(0) = 1$

- $\forall x \forall y \; E(x+y) = E(x) \cdot E(y)$

- $\forall y \, [\, y = 0 \vee \exists x \; E(x) = y \,]$

or

SK : standard kernel

$\ker E \cong \mathbb{Z}$

$\cancel{\forall x \, [\, E(x) = 1}$

$\cancel{\Leftrightarrow} \; \bigvee_{n \in \mathbb{Z}}$

$\cancel{\forall x \,[E(x) =}$

$\exists \varpi \,[\forall x \quad E(x) = 1$

$$\rightarrow \bigvee_{n \in \mathbb{Z}} \overbrace{\varpi + \cdots + \varpi}^{n\text{-times}} = x \,]$$

SC Schanuel's Conjecture

$\forall \alpha_1 \ldots \forall \alpha_n$

$[$ if $\dim_{\mathbb{Q}} (\mathbb{Q}\alpha_1 + \cdots + \mathbb{Q}\alpha_n) = n,$

then $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\vec{\alpha}, \overrightarrow{E(\alpha)})$

$\geq n$]

<u>Example</u> if ϖ has

$E(\varpi) = 1$, $\varpi \neq 0$

Then ϖ is transcendental

$\alpha_1 = \varpi$

$\operatorname{tr\,deg}_{\mathbb{Q}} \mathbb{Q}(\varpi, E(\varpi)) \geq 1$

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(\alpha, E(\alpha)) \geq n$$

$$\Updownarrow$$

$$\bigwedge \neg \big(f_1(\alpha, E\alpha) = 0 \ \& \ \cdots \ \& \ f_m(\alpha, E(\alpha)) = 0 \big)$$

$f_1, \ldots, f_m \in \mathbb{Q}[x_1, \ldots x_n, y_1, \ldots y_n]$

$\dim(V(\vec{f})) < n$

W1Eac

Schanuel's

Reverse Conjecture:

If $\mathcal{K} = (K, +, \cdot, E, 0, 1)$

is a countable

field satisfying ELA + SK

+ SC, then

$\exists$ embedding

$K \hookrightarrow \mathbb{C}_{exp}$

If $X \subseteq \mathbb{G}_a^g \times \mathbb{G}_m^g$

additive group multiplicative

is an algebraic variety (irreducible) & X is rotund

and additively & multiplicatively free

then $\exists\ a = (a_1, \ldots a_g)$

$\in \mathbb{G}_a^g(K)$

$(a, E(a)) \in X(K)$

—

additively free:

$\nexists\ (l_1, \ldots l_g) \in \mathbb{Z}^g \setminus \{(0, \ldots 0)\}$

$l_1 x_1 + \ldots + l x_g$ is constant

on X

multiplicatively free:

no $(l_1 \ldots l_g) \in \mathbb{Z}^g \setminus \{(0 \ldots 0)\}$

$y_1^{l_1} \cdots y_g^{l_g}$ is const

on X.

If $M = (M_{ij})$

$\in M_{g \times m}(\mathbb{Z})$

$\Phi_M : \mathbb{G}_a^g + \mathbb{G}_m^g \to \mathbb{G}_a^m + \mathbb{G}_m^m$

$(x_1 \ldots x_g, y_1 \ldots y_g)$

$\mapsto (\sum M_{1j} x_j, \ldots, \sum M_{mg} x_j,$

$\prod y_j^{M_{1j}}, \ldots, \prod y_j^{M_{mj}})$.

noted:

dim $\Phi_M X \geq$ rk M.

CCP: countable

closure.