ARITHMETIC DYNAMICS AND (UNLIKELY) INTERSECTIONS

PROJECT DESCRIPTIONS

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These two projects are centered around a general question that could be formulated, very roughly, as:

Question. In the dynamical moduli spaces of maps $f : \mathbb{P}^N \to \mathbb{P}^N$, or in the total spaces for families of such maps – for example looking at

$$F : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$$

for a complex manifold Λ, defined by $F(\lambda, z) = (\lambda, f_\lambda(z))$ and a family f_λ of maps on $\mathbb{P}^N$ – which phenomena are likely and which phenomena are unlikely?

Definitions and background on the dynamical moduli spaces M_d^N of maps $f : \mathbb{P}^N \to \mathbb{P}^N$ can be found in Silverman's lecture notes [Si3]. Briefly, two morphisms $f, g : \mathbb{P}^N \to \mathbb{P}^N$ are *equivalent* (over $\mathbb{C}$) if there exists an automorphism $A \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{P}^N$ such that $A \circ f \circ A^{-1} = g$. (Conjugacy preserves dynamical features.) Note that any morphism $f : \mathbb{P}^N \to \mathbb{P}^N$ is defined by $N + 1$ homogeneous polynomials $f_0, \ldots, f_N$ in $N + 1$ variables, each of the same degree d, with their zero sets having empty intersection. The case $d = 1$ are the automorphisms of $\mathbb{P}^N$. We assume $d > 1$. Paramaterizing by the coefficients of these f_i, we may view the space of all endomorphisms of $\mathbb{P}^N$ as an open subset End_d^N of some large complex projective space. The moduli space M_d^N is the quotient of End_d^N by the conjugation action by automorphisms.

The questions below are instances of (or steps towards proofs of) unlikely/likely intersections, as inspired by the conjectures of Manin-Mumford, André-Oort, Zilber and Pink and many others; see [Za]. One of the topics is more arithmetic, one is more geometric. Each of the questions can take you in several different directions, depending on your background and interests.

1. Bounded height and equidistribution in dimension 1

Fix two families of maps $f_t, g_t : \mathbb{P}^1 \to \mathbb{P}^1$, defined over $\overline{\mathbb{Q}}$, parameterized by t in a quasiprojective algebraic curve C. In other words, we consider an algebraic curve C (defined over $\overline{\mathbb{Q}}$) in the product $\mathrm{End}_d^1 \times \mathrm{End}_e^1$ of spaces of endomorphisms of $\mathbb{P}^1$, with degrees $d, e > 1$. Fix two maps $a, b : C \to \mathbb{P}^1$, also defined over $\overline{\mathbb{Q}}$, which we view as "marked points".

Question 1.1. *Is the set of orbit collisions between the marked points,*

$$\left\{t \in C(\overline{\mathbb{Q}}) : \exists\, n, m \geq 0 \text{ such that } f_t^n(a(t)) = g_t^m(b(t))\right\},$$

a set of bounded height on C?

Special cases and more general questions appear in the article [DGKNTY]; Conjecture 1.5 of that article suggests when this set should have bounded height. The question was inspired by the work of Bombieri-Masser-Zannier [BMZ1, BMZ2]. Bounded height is known in settings where both f and g are Lattès maps, as a consequence of "Silverman specialization" [Si1], and where f and g are both monomials [AMZ].

Project 1.2. *Extend Theorems 1.1 and 7.1 from* [DGKNTY] *to treat $f_t(z) = g_t(z) = z^d + t$ for arbitrary d and all pairs $a, b \in \overline{\mathbb{Q}}$ with $a^d \neq b^d$.*

Begin by reading the Introduction to [DGKNTY]. The outline of the proofs is provided at the end of the Introduction, and there are two main steps. A more detailed description of the proof is given in Section 5.1. Related simplifications are proved in Section 5, but the machinery of Section 6 was needed for the main proofs.

Project 1.3. *Focus on special cases of Question 1.1 where f and g are independent of the parameter t. For example, what can we say about the set*

$$\left\{z \in \overline{\mathbb{Q}} : \exists\, n, m \geq 1 \text{ such that } f^n(z) = g^m(z)\right\}$$

where $f(z) = z^2$ and $g(z) = z^2 - 1$? If we restrict to the subset where $n - m$ is held constant, do the solutions equidistribute to some natural measure as $n \to \infty$? See Figure 1.1.

For the example of Figure 1.1, the polynomials $P_n(z) = f^n(z) - g^n(z)$ appear to be "mostly" irreducible over $\mathbb{Q}$; that is, there is a factor of $z^{2^{n/2}}$ when n is even but might otherwise be irreducible. (I only tested to $n = 8$.)

The nonzero roots of P_n appear to equidistribute as $n \to \infty$ to the measure $\mu = \Delta \max\{G_f, G_g\}$. For any polynomial of degree $d > 1$, we define

$$G_f(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ = \max\{\log, 0\}$. This function is called the *escape-rate function* of f, and it coincides with the Green's function for the filled Julia set of f, with log-pole at ∞; see [Mi, p.100]. It is also the archimedean component of the *canonical height function* for f,

$$\hat{h}_f(\alpha) = \lim_{n \to \infty} \frac{1}{d^n} h(f^n(\alpha))$$

for all $\alpha \in \overline{\mathbb{Q}}$, where h is the standard logarithmic Weil height. See [Si2] for background on these canonical height functions. Equidistribution with respect to μ would

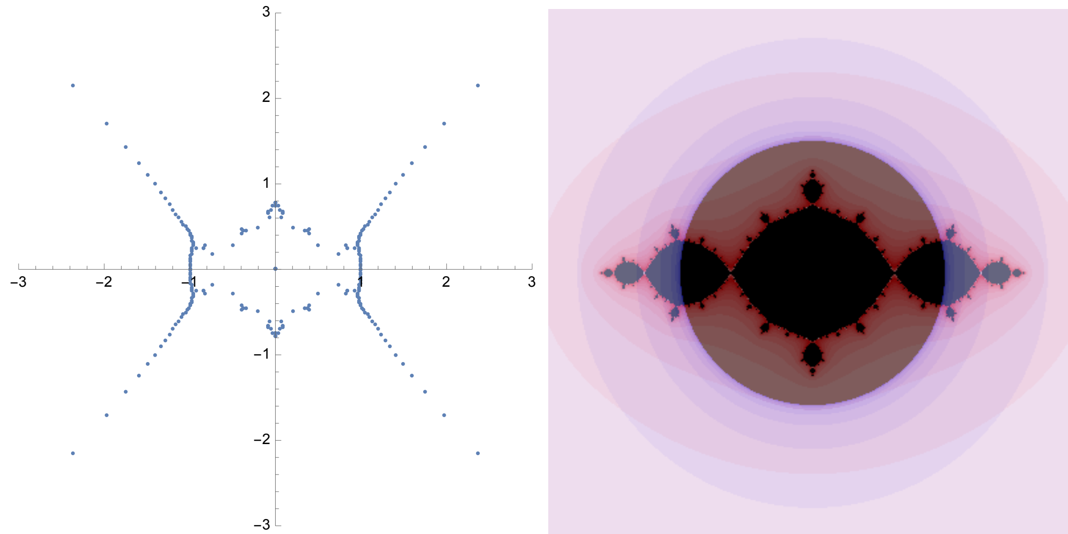

FIGURE 1.1. Left: solutions to $f^8(z) = g^8(z)$ for $f(z) = z^2$ and $g(z) = z^2 - 1$. Right: the filled Julia sets of f and g and level curves of G_f and G_g, superimposed.

mean that the discrete measures

$$\frac{1}{2^n} \sum_{f^n(z)=g^n(z)} \delta_z$$

in $\mathbb{P}^1(\mathbb{C})$ converge weakly to the measure μ as $n \to \infty$.

But do we have equidistribution in this setting? The existing equidistribution theorems for heights on $\mathbb{P}^1(\overline{\mathbb{Q}})$, for example of Baker-Rumely, Favre-Rivera-Letelier, or Chambert-Loir, (see, for example, [BR]) may not apply, or do they?

2. Invariant subvarieties in $\mathbb{P}^N$

Given a morphism $f : \mathbb{P}^2 \to \mathbb{P}^2$, we say it has an *invariant curve* if there exists a (possibly reducible) algebraic curve C in $\mathbb{P}^2$ so that $f(C) \subset C$. For example, the map

$$f_0(x : y : z) = (y^2 : z^2 : x^2)$$

of degree 2 permutes the components of $\{xyz = 0\}$. In fact, since $f_0^3(x : y : z) = (x^8 : y^8 : z^8)$ has such a simple form, we can find many invariant (unions of) lines in $\mathbb{P}^2$.

Question 2.1. *In the moduli space M_d^2, for $d \geq 2$, determine if the set of maps having an invariant curve is Zariski dense.*

Question 2.2. *More generally, what about the subset of maps in the moduli space M_d^N having an invariant subvariety (of dimension bigger than 0 and smaller than N) in $\mathbb{P}^N$?*

This question arose at the Simons Symposium in Complex and Algebraic Dynamics, August 2022. Below I provide a collection of background/related results. Studying the proofs in any of these papers could lead to a new result in this direction.

Fakhruddin proved [Fa, Theorem 1.2] that the *generic* endomorphism of $\mathbb{P}^N$ has no invariant subvarieties, other than points and all of $\mathbb{P}^N$. This means that, working over the function field of End_d^N, the "universal" map has no (non-trivial) invariant subvariety. Fakhruddin points out that this implies – over $\mathbb{C}$ – that the maps with invariant subvarieties lie in a countable union of hypersurfaces of End_d^N. Question 2.2 is then asking: do we need a countable union?

Project 2.3. *Consider first a special family of maps, more amenable to computation. For example, what happens for maps of the form $A \circ f_0$, as A runs through all possible* automorphisms *of $\mathbb{P}^2$?*

Much of the interest in Question 2.2 has been centered around a special case: If an invariant hypersurface $V \subset \mathbb{P}^N$ contains the critical locus of $f : \mathbb{P}^N \to \mathbb{P}^N$, where $\det Df$ vanishes, then the map is called *postcritically finite*. The example f_0 above is postcritically finite on $\mathbb{P}^2$. The subset of postcritically finite maps in M_d^N was studied recently in [IRS], where it was conjectured that such maps are *not* Zariski dense in M_d^N for any $N > 1$ and $d \geq 2$. Gauthier-Taflin-Vigny have recently announced a proof of this non-density result, but their preprint is not yet ready, though it should be available soon! (Note that the critical locus defines one particular hypersurface in $\mathbb{P}^N$, though of course it depends on the map f, while Question 2.2 asks about *any* subvariety.)

By contrast, the postcritically finite maps on $\mathbb{P}^1$ of a given degree $d \geq 2$ *are* Zariski dense in the moduli space M_d^1; my lecture notes from the 2015 "Komplex Analysis Winter School" [De2] include a proof of this statement and provide additional background. In fact, much more generally, for a family of maps on $\mathbb{P}^1$ defining a subvariety $V \subset \mathrm{M}_d^1$ of dimension ℓ, every collection of ℓ marked points will be simultaneously preperiodic for a Zariski dense set of parameters in V; this is proved in [De1], as a special case of the implication (2) $\implies$ (1) in Theorem 6.2, taking $a_0 = a_1$ in the list of marked points. For example, since $\dim \mathrm{M}_d^1 = 2d-2$, for any given collection of $2d-2$ points, $A = \{a_1, \ldots, a_{2d-2}\} \subset \mathbb{P}^1$, the forward orbit

$$A_f := \bigcup_{n \geq 0} f^n(A)$$

will be finite for a Zariski dense set of maps $f \in \mathrm{End}_d^1$, thus defining an invariant subvariety of $\mathbb{P}^1$ for these maps as $f(A_f) \subset A_f$. The proofs in [De1] all rely on Theorem 1.1 of that article; Gauthier and Vigny have generalized that result to families of maps on $\mathbb{P}^N$ in [GV, Theorem A]. This may lead to an approach to prove Zariski density in a more general setting.

On the other hand, for dynamics on $\mathbb{P}^N$ with $N > 1$ and in contrast to the postcritically finite maps, it is known that the postcritically *improper* maps on $\mathbb{P}^N$ are Zariski dense [Ol, Theorem 2]. In that article, Olechnowicz proves that the critical locus must contain periodic points for a Zariski dense set of maps. Contrast this to Fakhruddin's result that the generic endomorphism of $\mathbb{P}^N$ does not have any preperiodic points in its critical locus [Fa, Corollary 3.5]. As a step in the proof of his Theorem 2, Olechnowicz relies on the irreducibility of the critical locus in $\mathbb{P}^N$ for a general map f, but he points out that this is not known for degree 2 maps in dimensions ≥ 3.

Project 2.4. *Determine if the critical locus of $f : \mathbb{P}^N \to \mathbb{P}^N$ with degree $d = 2$ is irreducible for a Zariski-open set of maps in M_2^N for $N \geq 3$. Compare* [IRS, Theorems 14 and 15] *and* [Ol, §5 Remark 3].

Stepping away from the setting of the critical locus, it would be interesting to know, even in dimension $N = 2$, if there is a Zariski open subset of maps with no invariant curves at all. Another interesting variation on the theme, building on the more developed theory of 1-dimensional dynamics, is to study the case of product maps (f, g) acting on $\mathbb{P}^1 \times \mathbb{P}^1$. Pakovich presents a classification of invariant curves in [Pa].

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