ARITHMETIC DYNAMICS AND INTERSECTION PROBLEMS

LAURA DEMARCO

1. Overview

The goal of this lecture series is to present some complex-analytic and dynamical techniques that have been useful for studying algebraic and arithmetic intersection problems.

Lecture 1. The Lattès family. There is an important class of maps $f : \mathbb{P}^1 \to \mathbb{P}^1$, the *Lattès examples*, that has inspired many of the developments in arithmetic dynamical systems. These maps arise as the quotient of an endomorphism of an elliptic curve $\varphi : E \to E$. We begin by introducing these examples and presenting some of the parallels between the study of elliptic curves and the study of dynamics on $\mathbb{P}^1$. Helpful references: [Mi] [Si1].

Lecture 2. Pluripotential theory. We introduce key tools in the study of complex analysis and dynamics in dimensions > 1, namely the theory of currents and plurisubharmonic functions. Helpful reference: [Dem].

Lecture 3. Dynamical stability. We introduce the dynamical concept of structural stability for families of maps. We illustrate this concept in the setting of the Lattès family and other important examples on $\mathbb{P}^1$, and we relate stability to the values of certain (geometric) height functions. Helpful references: [De] [Mc, Chapter 4].

Lecture 4. Bifurcation measures and adelic line bundles. Working over number fields, we broaden the notion of dynamical stability into the general framework of the theory of adelic line bundles on quasiprojective varieties. We present examples from the recent work of Yuan-Zhang [YZ] and Gauthier-Vigny [GV].

2. Projects

These projects are centered around a general question that could be formulated, very roughly, as:

Question. *In the dynamical moduli spaces of maps $f : \mathbb{P}^N \to \mathbb{P}^N$, which phenomena are likely and which phenomena are unlikely?*

Definitions and background on these moduli spaces can be found in Silverman's lecture notes [Si2]. Briefly, two morphisms $f, g : \mathbb{P}^N \to \mathbb{P}^N$ are *equivalent* (over $\mathbb{C}$) if there exists an automorphism $A \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{P}^N$ such that $A \circ f \circ A^{-1} = g$. (Conjugacy preserves dynamical features.) We denote the space of all maps on $\mathbb{P}^N$ with degree $d \geq 2$, modulo this equivalence, by M_d^N.

The questions below are instances of (or steps towards proofs of) unlikely/likely intersections, as inspired by the conjectures of Manin-Mumford, André-Oort, Zilber and Pink and many others; see [Za]. One of the problems is more arithmetic, one is more geometric. I will provide more details before the Arizona Winter School begins. Hopefully this gives an idea of what I have in mind:

(1) **Bounded height.** Fix two families of maps $f_t, g_t : \mathbb{P}^1 \to \mathbb{P}^1$, defined over $\overline{\mathbb{Q}}$, parameterized by t in an algebraic curve C. Fix two maps $a, b : C \to \mathbb{P}^1$, also defined over $\overline{\mathbb{Q}}$. Is the set of orbit collisions

$$\left\{t \in C(\overline{\mathbb{Q}}) : \exists\, n, m \geq 0 \text{ such that } f_t^n(a(t)) = g_t^m(b(t))\right\}$$

of bounded height on C? Special cases and more general questions appear in [DGKNTY]. For example, what about the set

$$\left\{z \in \overline{\mathbb{Q}} : \exists\, n, m \geq 0 \text{ such that } f^n(z) = g^m(z)\right\}$$

where $f(z) = z^2$ and $g(z) = z^2 - 1$? Bounded height is known in settings where both f and g are monomials or Lattès maps. Compare [AMZ].

(2) **Invariant curves in $\mathbb{P}^2$.** Given a map $f : \mathbb{P}^2 \to \mathbb{P}^2$, we say it has an *invariant curve* if there exists a (possibly reducible) algebraic curve C in $\mathbb{P}^2$ so that $f(C) \subset C$. For example, the map

$$f_0(x : y : z) = (y^2 : z^2 : x^2)$$

of degree 2 permutes the components of $\{xyz = 0\}$. In fact, since $f_0^3(x : y : z) = (x^8 : y^8 : z^8)$ has such a simple form, we can find many invariant (unions of) lines in $\mathbb{P}^2$.

In the moduli space M_d^2, for $d \geq 2$, determine if the set of maps having an invariant curve is Zariski dense.

Remarks: If an invariant curve contains the critical locus of f, where $\det Df$ vanishes, then the map is called *postcritically finite*. My example f_0 above is postcritically finite. The subset of postcritically finite maps in M_d^2 has been studied in [IRS], where it is conjectured that such maps are *not* Zariski dense in M_d^N for any $N > 1$ and $d \geq 2$. By contrast, the postcritically finite maps on $\mathbb{P}^1$ of a given degree $d \geq 2$ *are* Zariski dense in the moduli space M_d^1 [De]. As are the postcritically "improper" maps on $\mathbb{P}^2$ [Ol]. But now we ask only about the existence of an invariant curve, with no additional restrictions. This question arose at the Simons Symposium in Complex and Algebraic Dynamics, August 2022.

References

[AMZ] F. Amoroso, D. Masser, and U. Zannier. Bounded height in pencils of finitely generated subgroups. *Duke Math. J.* **166**(2017), 2599–2642.

[Dem] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry.* Version of June 21, 2012.

[De] Laura DeMarco. Dynamical moduli spaces and elliptic curves (KAWA Lecture Notes). *Ann. Fac. Sci. Toulouse Math.* **27**(2018), 389–420.

[DGKNTY] Laura DeMarco, Dragos Ghioca, Holly Krieger, Khoa Dang Nguyen, Thomas Tucker, and Hexi Ye. Bounded height in families of dynamical systems. *Int. Math. Res. Not. IMRN* (2019), 2453–2482.

[GV] Thomas Gauthier and Gabriel Vigny. The geometric dynamical Northcott and Bogomolov properties. *Preprint,* arXiv:1912.07907v2 [math.DS].

[IRS] Patrick Ingram, Rohini Ramadas, and Joseph H. Silverman. Post-critically finite maps on $\mathbb{P}^n$ for $n \geq 2$ are sparse. *Preprint,* arXiv:1910:11290 [math.DS].

[Mc] Curtis T. McMullen. *Complex Dynamics and Renormalization.* Princeton University Press, Princeton, NJ, 1994.

[Mi] John Milnor. On Lattès maps. In *Dynamics on the Riemann sphere*, pages 9–43. Eur. Math. Soc., Zürich, 2006.

[Ol] Matt Olechnowicz. Dynamically improper hypersurfaces for endomorphisms of projective space. *Preprint,* arXiv:2206.10678v2 [math.DS].

[Si1] Joseph H. Silverman. *The Arithmetic of Dynamical Systems*, volume 241 of *Graduate Texts in Mathematics.* Springer, New York, 2007.

[Si2] Joseph H. Silverman. *Moduli spaces and arithmetic dynamics*, volume 30 of *CRM Monograph Series.* American Mathematical Society, Providence, RI, 2012.

[YZ] Xinyi Yuan and Shouwu Zhang. Adelic line bundles over quasiprojective varieties. *Preprint,* arXiv:2105.13587v3 [math.NT].

[Za] Umberto Zannier. *Some problems of unlikely intersections in arithmetic and geometry*, volume 181 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 2012. With appendixes by David Masser.

E-mail address: `demarco@math.harvard.edu`