Families of dynamical systems

Algebraic family of maps on $\mathbb{P}^N$

is $f : B \times \mathbb{P}^N \xrightarrow{\text{morphism}} B \times \mathbb{P}^N$

B = (quasiprojective) alg. curve / $\mathbb{C}$

$\forall b \in B(\mathbb{C})$, $f_b : \mathbb{P}^N \to \mathbb{P}^N$

$$f(b, z) = (b, f_b(z)).$$

Special points $\longleftrightarrow$ preperiodic points

Special subvariety $\longleftrightarrow$ preperiodic subvariety

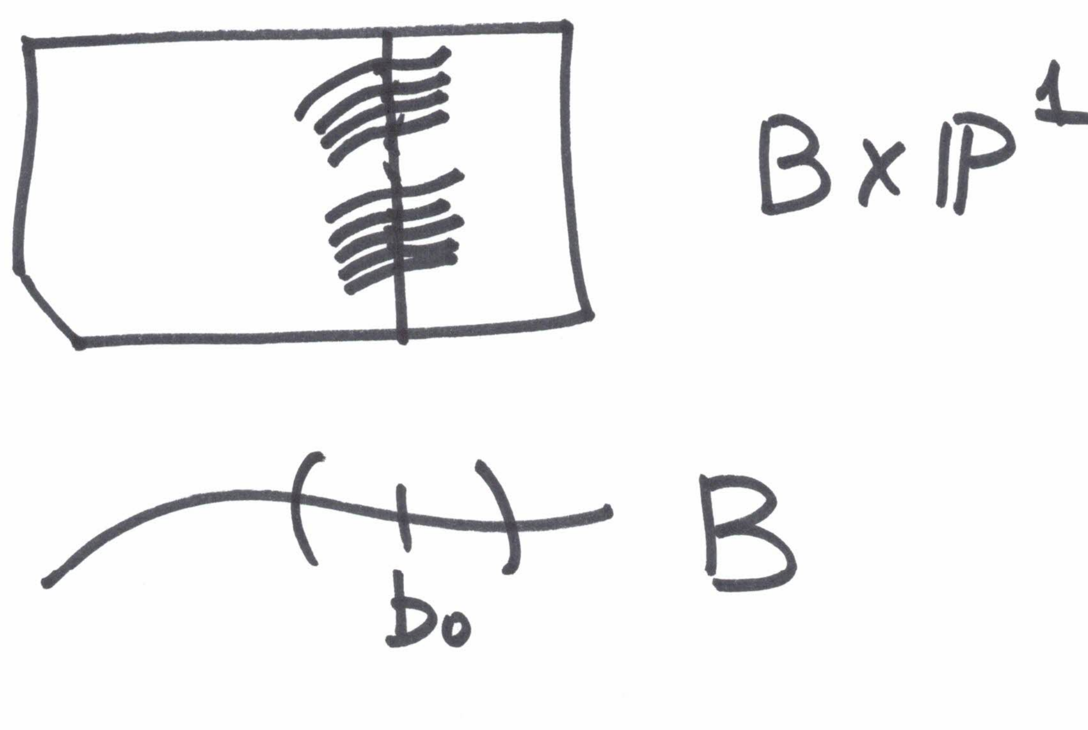
$f^n(V) = f^m(V)$

$\boxed{N=1}$ $f: B \times \mathbb{P}^1 \to B \times \mathbb{P}^1$

f is <u>periodic-point stable</u> at $b_0 \in B$ if

in a nbhd of b_0,

we can parameterize the periodic points and there are no collisions



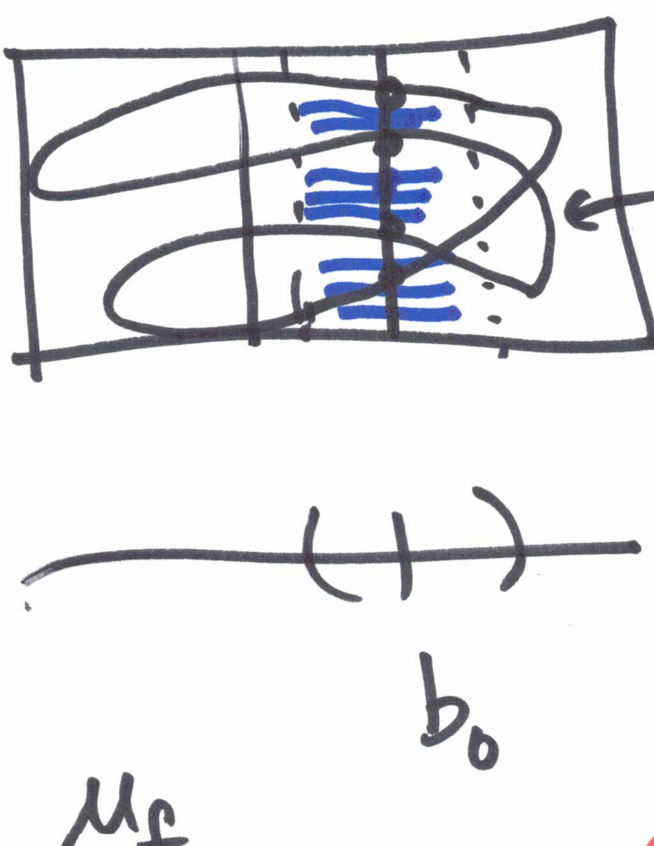
③

f is <u>critically stable</u> at $b_0 \in B$ if in a nbhd of b_0,

$$\hat{T}_f \wedge [\mathrm{Crit}(f)] = 0$$

current of integration

pos. (1,1)-current on $B \times \mathbb{P}^1$



Crit(f)

measure supported on Crit(f). detecting (deg d>1) intersections with Supp μ_f

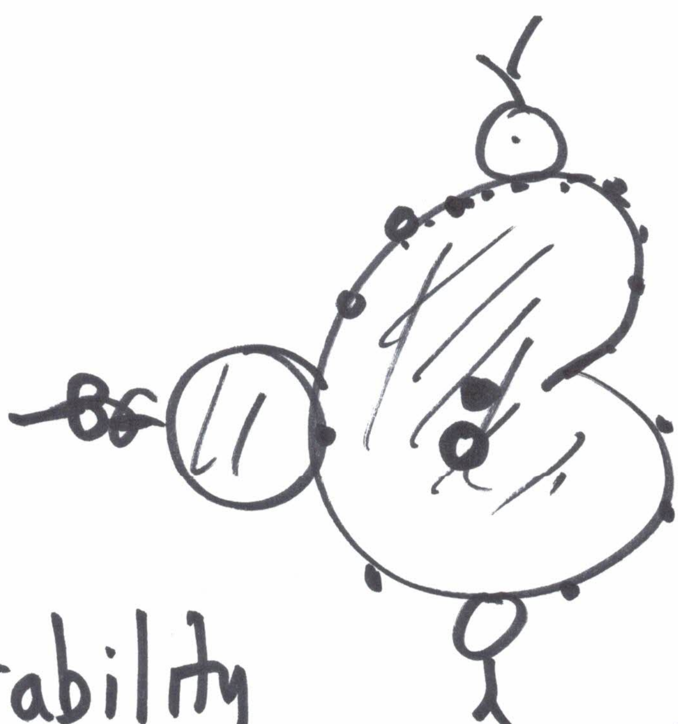
$$\hat{T}_f \Big|_{\{b\} \times \mathbb{P}^1} = \mu_f$$

<u>Thm</u> periodic · point stable at b_0 $\Longleftrightarrow$ critically stable at b_0.

Mañé - Sad - Sullivan, 1983
Lyubich 1983

<u>Example 1</u> $f_b(z) = z^2 + b$

$b \in \mathbb{C}$



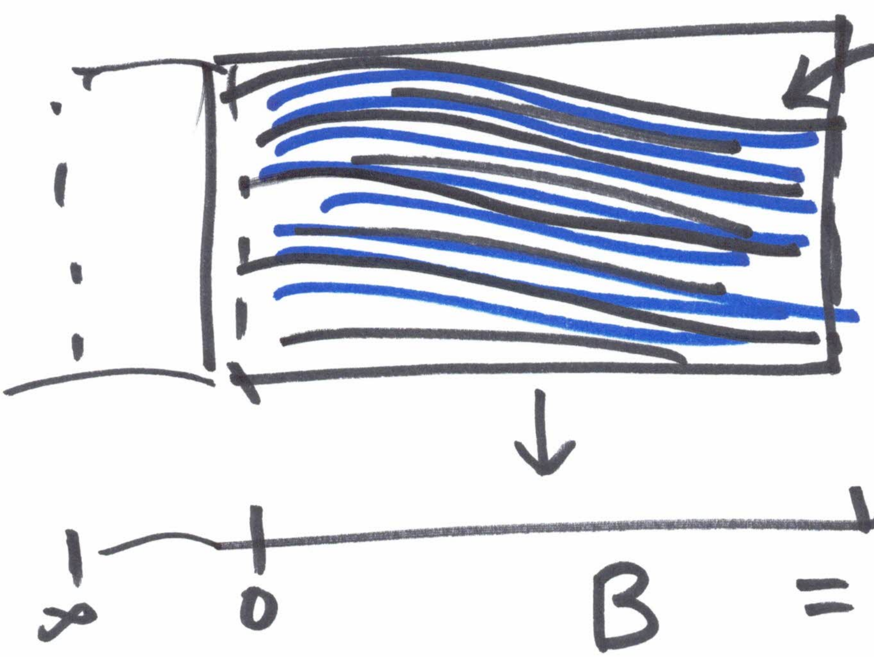
$\mathcal{M} = \{ b \in \mathbb{C} : J(f_b) = \operatorname{Supp} \mu_{f_b}$ is connected$\}$

Stability $\Longleftrightarrow b_0 \in \operatorname{Int} \mathcal{M} \cup (\mathbb{C} \setminus \mathcal{M})$.

Example 2 Lattès family

$$\begin{tikzcd} \mathcal{E} \arrow[loop left, "{[2]}"] \arrow[r] \arrow[d] & B \times \mathbb{P}^1 \arrow[loop right, "f"] \arrow[d] \\ B & B \end{tikzcd}$$

$$\begin{array}{c} E \\ \pi \downarrow \quad p \sim -p \\ \mathbb{P}^1 \end{array}$$



Critical values of f.

$= \pi(E[2])$

$B = \underline{\mathbb{C} \setminus \{0,1\}}$

$$f_b(z) = \frac{(z^2 - b)^2}{4z(z-1)(z-b)}$$

<u>stable</u> on all of B.

<u>Theorem</u> (McMullen, 1987)

If $f: B \times \mathbb{P}^1 \to B \times \mathbb{P}^1$ <u>alg.</u> family of maps, and if f is stable on <u>all</u> of B, then

- f is isotrivial

OR

- f is a Lattès family.

<u>Thm</u> (MSS, L) Stability is open & dense in B.

<u>Theorem</u> (Dujardin–Favre
D.)

$f : B \times \mathbb{P}^1 \to B \times \mathbb{P}^1$ alg. family.

Assum f is <u>not</u> isotrivial.

Take any alg. curve

$$C \subset B \times \mathbb{P}^1$$

Then $\hat{T}_f \wedge [C] = 0$

iff C is a preperiodic curve.

$f^n(C) = f^m(C)$

Consequence (already well known):

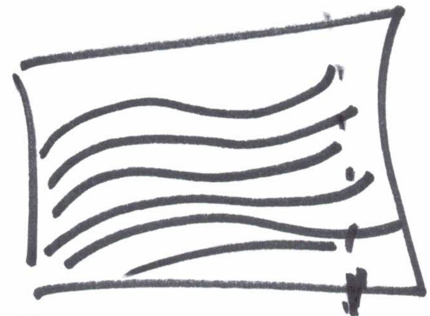
(smooth fibers)

$\mathcal{E}$ elliptic surface over B.

quasi proj

$$\begin{array}{c}\mathcal{E}\\ \downarrow\\ B\end{array}$$

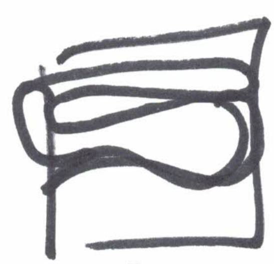
~~Betti~~ Betti form $= \omega$



Betti foliation.

The only algebraic leaves of this foliation are the torsion leaves.

$\hat{T}_f$



$\downarrow$

$B \times \mathbb{P}^1$

$b \mapsto c(b)$

$\mathcal{F}_c := \{ b \longmapsto f_b^n(c(b)) \}$

<u>Stability</u> $\iff$ $\mathcal{F}_c$ is a normal family.

$\boxed{\text{Montel}}$