AUTOMORPHIC FORMS AND THE THETA CORRESPONDENCE

WEE TECK GAN

1. Course Outline

This course gives a quick introduction to the technique of theta correspondence, as a means of constructing interesting examples of automorphic forms, with special focus on automorphic forms on unitary groups. Theta correspondence was used by Maass and others but was formally organized into a theory by Roger Howe in the 1970's, with the Weil representation and theta functions playing a central role. It complements the technique of trace formulas as a powerful tool for understanding representations and automorphic forms of classical groups. As an example, using theta correspondence, Howe and Piatetski-Shapiro constructed the first counterexamples to the generalized Ramanujan conjecture, and this is where we will begin the course.

The course will comprise of 4 lectures, tentatively devoted to the following topics:

(1) In the first lecture, we will recall the classical Ramanujan conjecture on Fourier coefficients of cusp forms, and reformulate it as a representation theoretic statement for cuspidal automorphic representations, namely that all cuspidal representations are tempered. We will then describe the construction of Howe-PS which give a counterexample to this conjecture for the group U(3), introducing the global and local theta lifting in the process.

(2) In the second lecture, we will introduce the theory of local theta correspondence more formally, describing the main actors (dual pairs and Weil representations) and highlighting the main questions (nonvanishing, irreducibility and determination of theta lifts). We will only have time to formulate answers to some of these questions and show some sample computations.

(3) In the third lecture, we will consider global theta correspondence and discuss the global analog of the local questions from Lecture 2. In particular, we will discuss the questions of checking the cuspidality and nonvanishing of global theta lifts.

(4) In the final lecture, we will place the example of Howe-PS in the context of Arthur's conjecture which classifies square-integrable automorphic forms. In particular, Arthur's conjecture quantifies the failure of Ramanujan's conjecture by giving a classification of nontempered cusp forms, After formulating Arthur's conjectures (in particular the notion of A-parameters, A-packets and the Arthur multiplicity formula), we will work out some other low rank examples, such as Saito-Kurokawa forms on PGSp(4) and the families of nontempered forms on the exceptional group G_2. We will see how the theta correspondence allows us to construct some of these A-packets. If time permits (most probably it won't), we will mention some recent results of Rui Chen and

Jialiang Zou on using theta correspondence to deduce the Arthur conjecture for non-quasi-split unitary groups from the case of quasi-split groups established by Arthur via the trace formula.

Background reading:

- W.T. Gan, Automorphic forms and automorphic representations. Automorphic forms and the Langlands program, 68-134, Adv. Lect. Math. (ALM), 9, Int. Press, Somerville, MA, 2010. This explains the transition from the theory of classical modular forms to the language of automorphic representations, as well as the reformulation of Ramanujan's conjecture to the representation theoretic setting.
- W.T. Gan, Theta correspondence: recent progress and applications. Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, 343-366. This provides a brief account of the theory of theta correspondence, the basic questions it addresses, the progress till 2014 and some applications.
- W.T. Gan and N. Gurevich, Non-tempered Arthur packets of G_2. Automorphic representations, L-functions and applications: progress and prospects, 129-155, Ohio State Univ. Math. Res. Inst. Publ., 11, de Gruyter, Berlin, 2005. This is a survey on the Arthur conjecture and the working out of its implications for G_2. It will be a good starting point for Project 1 below.

2. Projects

Here are some projects I have in mind:

- **Project 1**: Constructing nontempered A-packets of G_2.

  In the lecture, we will give as an example the various families of nontempered cusp forms on G_2. All but one of these families have been constructed by using theta correspondence in exceptional groups. A subclass of the last remaining family should be constructible using the theta correspondence for $\mathrm{U}_3 \times G_2$ with the Howe-PS cusp forms as inputs. This is a multi-step project which involves:

  - understanding and elucidating what the Arthur's conjecture says for this particular family of nontempered cusp forms on G_2;
  - formulating more precisely the above strategy for constructing these cusp forms by theta correspondence
  - establishing certain properties of the Howe-PS forms necessary to carry out the above proposed construction, e.g. understanding the torus periods of the Howe-PS forms.
  - understanding and refining the results of local theta lifting for $\mathrm{U}_3 \times G_2$ (this is some ongoing work of Bakić and Savin, hopefully ready by the time of AWS);
  - showing the nonvanishing of the global theta lifts by computing their Fourier expansion on G_2 (this is related to Pollack's lectures)
  - showing that the constructed forms agree with what Arthur's conjecture predicts, in particular establihsing the Arthur multiplciity formula for them.

Given the number of steps involved, this could be broken down into 2 or 3 project groups.

- **Project 2:** Variants of the Siegel-Weil formula.

  The Siegel-Weil formula in the context of unitary groups identifies the global theta lift of the trivial representation of $\mathrm{U}(m)$ to the group $\mathrm{U}(n,n)$ with a Siegel Eisenstein series of $\mathrm{U}(n,n)$. This projects considers the theta lift of the trivial representation of $\mathrm{U}(m)$ to $\mathrm{U}(n+1,n)$. One can ask if this theta lift, which is a rather degenerate automorphic form on $\mathrm{U}(n+1,n)$, can be described in an alternative way, e.g. as an Eisenstein series. For this, one would first need to understand the local setting: what is the local theta lift of the trivial representation? Such questions have been addressed in recent works of Hanzer and Bakić. I don't know if there is a reasonably nice answer or what to expect.

- **Project 3:** Verifying the global twisted GGP conjecture in low rank.

  Gross, Prasad and I recently formulated a twisted version of the GGP conjectures in the basic Fourier-Jacobi case. As a check, we verified our local conjectures in low rank, i.e. for $\mathrm{U}(n)$ for $n \leq 2$. One can likewise verify the global (refined) conjecture in low rank, where it reduces to some previously known results. Using theta correspondence, one can attempt to verify endoscopic cases of the global (refined) conjecture in the setting of $\mathrm{U}(3)$. This is similar to what Ichino and I did for the GP conjecture for $\mathrm{SO}(5) \times \mathrm{SO}(4)$.

We will have Petar Bakić as our project assistant. As you can see from the above description, his work is intimately connected with Projects 1 and 2 above.