AWS 2021: Modular Groups
Problem Set 5

Lecturer: Lori Watson
Written by: Tyler Genao, Hyun Jong Kim, Zonia Menendez and Sam Mundy (Assistants)

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1 Definitions and Notations

1. Given two compact Riemann surfaces X and Y, a nonconstant holomorphic map

$$f : X \to Y$$

has a well-defined *degree* $\deg(f)$. One has for all but finitely many $y \in Y$ that $\#f^{-1}(y) = \deg(f)$.

2. One has a relation between the degree of a nonconstant holomorphic map and its ramification indices,

$$\sum_{x \in f^{-1}(y)} e_x = \deg(f).$$

It follows that for $y \in Y$, one has $\#f^{-1}(y) = \deg(f)$ iff each $x \in f^{-1}(y)$ is *unramified,* i.e., $e_x = 1$.

3. Let us write $X(N) := X(\Gamma(N))$, $X_1(N) := X(\Gamma_1(N))$ and $X_0(N) := X(\Gamma_0(N))$.

4. From here on out, we will write $\Gamma(1) := \mathrm{SL}_2(\mathbb{Z})$.

5. Given two congruence subgroups $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma(1)$, one has a natural map of modular curves,

$$\pi : X(\Gamma_1) \to X(\Gamma_2)$$

via

$$\Gamma_1\tau \mapsto \Gamma_2\tau.$$

This is a nonconstant holomorphic map between compact Riemann surfaces.

6. For a congruence subgroup $\Gamma \subseteq \Gamma(1)$, a special case of the above is the natural map

$$X(\Gamma) \to X(1)$$

via

$$\Gamma\tau \mapsto \Gamma(1)\tau.$$

We sometimes call this map the *j-line map,* and $X(1)$ the *j-line.*

7. For congruence subgroups $\Gamma_1 \subseteq \Gamma_2$, let $\pi : X(\Gamma_1) \to X(\Gamma_2)$ be the natural projection map, $\Gamma_1\tau \mapsto \Gamma_2\tau$. Then one can compute ramification indices of points over π as follows. If $x := \Gamma_1\tau$ is a point on $X(\Gamma_1)$, then its ramification index is

$$e_x = [\{\pm I\}\mathrm{Stab}_{\Gamma_2}(\tau) : \{\pm I\}\mathrm{Stab}_{\Gamma_1}(\tau)].$$

For a proof of this fact, see Section 3.1 of Diamond & Shurman.

8. In the context of congruence subgroups, a point $\tau \in \mathcal{H}$ is called an *elliptic point for* Γ if its stabilizer is nontrivial, i.e., $\mathrm{Stab}_\Gamma(\tau) \supsetneq \{\pm I\}$. Its corresponding point $\Gamma\tau \in X(\Gamma)$ is also called an *elliptic point.* Compare this to the definition of elliptic points for Fuchsian subgroups in the previous problem set.

9. Given an elliptic point $\Gamma\tau \in X(\Gamma)$, its *period* is the index

$$h_{\Gamma\tau} := [\{\pm I\}\mathrm{Stab}_\Gamma(\tau) : \{\pm I\}] = \begin{cases} \#\mathrm{Stab}_\Gamma(\tau)/2 & \text{if } -I \in \Gamma \\ \#\mathrm{Stab}_\Gamma(\tau) & \text{if } -I \notin \Gamma \end{cases}.$$

The following definitions concern divisors on compact Riemann surfaces, see Problems 8, 15, 16, 17 and 19. Throughout, we let X denote a compact Riemann surface.

10. A *divisor* on X is a formal (finite) sum $D = \sum n_i P_i$ where $n_i \in \mathbb{Z}$ and $P_i \in X$. In particular, the group of divisors $\mathrm{Div}(X)$ is an abelian group generated by the points of X.

11. A divisor D is *effective* if all its coefficients are non-negative. We write $D \geq 0$ for an effective divisor.

12. The *degree* of a divisor D is the sum of its coefficients, i.e.,

$$\deg(D) := \sum n_i.$$

13. A *meromorphic function* on X is a holomorphic function f on the complement $X \setminus \Xi$ of some discrete subset Ξ of U that has at worst a pole at each point of Ξ.

    In terms of coordinate neighborhoods: if (U, φ) is a coordinate neighborhood of X, then the map $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is a meromorphic function on $\varphi(U)$[1]. In particular, around a point $a \in \varphi(U)$, there is some non-negative integer n such that $(z-a)^n \cdot (f \circ \varphi^{-1})(z)$ is holomorphic at $z = a$.

14. For a nonzero meromorphic function f on X, for each $P \in X$ let us define the *order of* f *at* P. We write it as $\mathrm{ord}_P(f) = m, -m$, or 0, according to whether f has a zero of order m at P, a pole of order m at P, or neither a pole nor a zero at P. The *divisor* of f is then

$$\mathrm{div}(f) := \sum_{P \in X} \mathrm{ord}_P(f) \cdot P$$

    (This is a finite sum since the X is compact and the zeros and poles of f form discrete sets.) Such a divisor $\mathrm{div}(f)$ is called a *principal divisor* of X. One can show that a principal divisor has degree 0.

15. Two divisors D_1, D_2 on X are called *linearly equivalent* if their difference $D_1 - D_2$ is principal, i.e., $D_1 - D_2 = \mathrm{div}(f)$ for some nonzero meromorphic function f on X.

16. For a divisor D, we can define its *Riemann Roch space* as

$$L(D) := \{\text{nonzero meromorphic } f \text{ on } X : \mathrm{div}(f) + D \geq 0\} \cup \{0\}.$$

    This is a $\mathbb{C}$-vector space; denote its dimension by $l(D)$.

2 Introductory Problems

A note: to prove genus formulas for specific modular curves like $X_0(\ell), X_1(\ell)$ and $X(\ell)$, follow Problems $1 \to 9 \to 10 \to 11 \to 12 \to 13 \to 18$; you should also try Problem 3.

Problem 1.

a. Show that for congruence subgroups $\Gamma_1 \subseteq \Gamma_2$, the cusps of $X(\Gamma_1)$ are precisely the preimages of the cusps of $X(\Gamma_2)$ under the natural projection map.

[1] $\varphi(U)$ is an open subset of $\mathbb{C}$.

b. Show that $X(\Gamma)$ has finitely many cusps for any congruence subgroup Γ.

Problem 2. Show that for any congruence subgroup $\Gamma \subseteq \Gamma(1)$, $X(\Gamma)$ has finitely many elliptic points.

Problem 3. Recall that a compact Riemann surface with genus $g = 1$ is a complex elliptic curve, i.e., a complex torus.

Using the genus formulas in Problem 18, create a list of all prime numbers $\ell \geq 5$ for which $X(\ell)$ is a complex elliptic curve. Do the same for $X_1(\ell)$ and $X_0(\ell)$.

Problem 4. Show that each noncuspidal point $\Gamma(1)\tau \in X(1)$ has a well-defined *j-invariant* $j(\Gamma\tau) := j([1,\tau])$.

Problem 5. Let $\Gamma \subseteq \Gamma(1)$ be a congruence subgroup.

a. Show that the natural map $X(\Gamma) \to X(1)$ can only ramify over the points $\Gamma(1)i, \Gamma(1)\zeta_3$ and $\Gamma(1)\infty$. (*Hint:* see Definition 7.)

b. Determine explicitly which noncuspidal points are ramified under $X_1(N) \to X(1)$. Do the same analysis with the map $X_0(N) \to X(1)$.

Problem 6. What is the moduli space interpretation of the natural map $X_1(N) \to X_0(N)$ on the noncuspidal points? What about the map $X_0(N) \to X(1)$?

Problem 7. Let $\Gamma \subset \Gamma(1)$ be a congruence subgroup. By Problem 5, we know that $\Gamma(1)i$ and $\Gamma(1)\zeta_3$ are the only noncuspidal points which can ramify under the map $X(\Gamma) \to X(1)$. Similar to Problem 6, we can interpret this map in terms of elliptic curves. How is this ramification related to elliptic curves corresponding to the points $\Gamma(1)i$ and $\Gamma(1)\zeta_3$ in $X(1)$?

Problem 8. Let X be a compact Riemann surface.

a. Given nonzero meromorphic functions f and g on X, show that $\mathrm{div}(f \cdot g) = \mathrm{div}(f) + \mathrm{div}(g)$. Also show that $\mathrm{div}(f^{-1}) = -\mathrm{div}(f)$.

b. A constant c on X can be interpreted as a meromorphic function on X. Show that $\mathrm{div}(c) = 0$.

c. Say that f is a nonzero meromorphic function such that $\mathrm{div}(f)$ is effective. Show that f is holomorphic.

3 Intermediate Problems

Problem 9 (Diamond & Shurman, Exercise 2.3.7)**.** Show there are no elliptic points for the following congruence subgroups.

a. $\Gamma(N)$ for $N > 1$;

b. $\Gamma_1(N)$ for $N > 3$;

c. $\Gamma_0(N)$ for any N divisible by some prime congruent to $-1 \pmod{12}$.

Problem 10 (Degrees of maps between modular curves)**.** Show that for $\Gamma_1 \subseteq \Gamma_2$, the projection map $\pi : X(\Gamma_1) \to X(\Gamma_2)$ satisfies

$$\deg(\pi) = \begin{cases} [\Gamma_2 : \Gamma_1]/2 & \text{if } -I \in \Gamma_2 \setminus \Gamma_1 \\ [\Gamma_2 : \Gamma_1] & \text{else} \end{cases}.$$

(*Hint:* Choose a set of coset representatives for $\{\pm I\}\Gamma_1$ in $\{\pm I\}\Gamma_2$. Then for infinitely many non-elliptic points $\Gamma_2\tau \in X(\Gamma_2)$, determine the size $\#\pi^{-1}(\Gamma_2\tau)$.)

Problem 11. Following Problem 10, we can determine the degrees of the natural maps between various modular curves.

Assume that $N \geq 3$.

a. Show that

$$\deg(X_0(N) \to X(1)) = \psi(N)$$

where $\psi(N) := N \prod_{p|N}(1 + 1/p)$ is the *Dedekind psi function.*

b. Show that

$$\deg(X_1(N) \to X(1)) = \frac{\phi(N)\psi(N)}{2}$$

where $\phi(N) := N \prod_{p|N}(1 - 1/p)$ is *Euler's totient function.*

c. Show that

$$\deg(X(N)) \to X(1)) = \frac{N\phi(N)\psi(N)}{2}.$$

d. Show that

$$\deg(X_1(N) \to X_0(N)) = \frac{\phi(N)}{2}.$$

(*Hint:* refer to Problem Set 2 for each relevant index.)

Problem 12 (Diamond & Shurman, Exercise 3.1.5)**.** This exercise will show that for an odd prime $\ell \in \mathbb{Z}^+$, $X_1(\ell)$ has exactly $\ell - 1$ cusps.

a. Show that any element $\gamma \in \mathrm{Stab}_{\Gamma(1)}(s)$ has trace ± 2.

b. Show that for any cusp $s \in \mathbb{P}^1(\mathbb{Q})$, one has $\mathrm{Stab}_{\Gamma_0(\ell)}(s) = \mathrm{Stab}_{\{\pm I\}\Gamma_1(\ell)}(s)$.

c. Conclude that the natural map $X_0(\ell) \to X_1(\ell)$ is unramified at the cusps.

d. Use Problems 1 and 11 and Definitions 2 and 7 to conclude that $X_1(\ell)$ has exactly $\ell - 1$ cusps for odd primes $\ell \in \mathbb{Z}^+$.

Problem 13 (Diamond & Shurman, Exercise 3.1.4)**.** Let $\ell \in \mathbb{Z}^+$ be prime.

1. Show that the number of elliptic points of period 2 in $X_0(\ell)$ is the number of solutions to $x^2 + 1$ mod ℓ, which is 2 if $\ell \equiv 1 \pmod 4$, 0 if $\ell \equiv 3 \pmod 4$ and 1 if $\ell = 2$.
2. Show that the number of elliptic points of period 3 in $X_0(\ell)$ is the number of solutions to $x^2 - x + 1$ mod ℓ, which is 2 if $\ell \equiv 1 \pmod 3$, 0 if $\ell \equiv 2 \pmod 3$ and 1 if $\ell = 3$.

(*Hint:* Use the coset decomposition

$$\Gamma(1) = \Gamma_0(\ell)\alpha_\infty \cup \bigcup_{j=0}^{\ell-1} \Gamma_0(\ell)\alpha_j$$

where $\alpha_j := \begin{bmatrix} 1 & 0 \\ j & 1 \end{bmatrix}$ and $\alpha_\infty := \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ to deduce that the elliptic points of Γ are contained in the subset $\{\Gamma\alpha_j \cdot i, \Gamma\alpha_j \cdot \zeta_3 : 0 \leq j \leq \ell - 1\} \cup \{\Gamma\alpha_\infty \cdot i, \Gamma\alpha_\infty \cdot \zeta_3\}$ of $X(\Gamma)$.)

Problem 14 (Diamond & Shurman, Exercise 1.5.4)**.** For nonzero integer $N \in \mathbb{Z}^+$, consider the matrix $\omega_N := \begin{bmatrix} 0 & -1 \\ N & 0 \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$.

a. Show that ω_N normalizes $\Gamma_0(N)$, and thus gives an automorphism $\Gamma_0\tau \mapsto \Gamma_0\omega_N \cdot \tau$ of the modular curve $X_0(N)$.

b. Show that this automorphism is an *involution* (i.e., has order 2).

c. Regarding $X_0(N)$ as a moduli space, describe the corresponding automorphism on the noncuspidal points of $X_0(N)$.

Problem 15. Let X be a compact Riemann surface. Let D be a divisor with negative degree. Show that $l(D) = 0$. (*Hint:* A principal divisor has degree 0.)

Problem 16. Let X be a compact Riemann surface.

a. Show that if f is a nonzero meromorphic function on X with $\mathrm{div}(f) = 0$, then f is constant.

b. Show that if f, g are nonzero meromorphic functions on X with $\mathrm{div}(f) = \mathrm{div}(g)$, then f is a constant multiple of g.

c. Show that $l(0) = 1$.

Problem 17 (Divisors and the Riemann-Roch theorem)**.** The Riemann-Roch theorem for compact Riemann surfaces,[2] which we do not prove, states the following: Let X be a compact Riemann surface with genus g. Then there is a divisor K, called the *canonical divisor of* X,[3] such that for every divisor D of X one has

$$l(D) - l(K - D) = \deg(D) + 1 - g.$$

a. Show that $\deg(K) = 2g - 2$ and $l(K) = g$.

b. If $\deg(D) > 2g - 2$, then show that Riemann-Roch simplifies: $l(D) = \deg(D) + 1 - g$.

4 Advanced Problems

Problem 18 (The genus of a modular curve)**.** Given a congruence subgroup $\Gamma \subseteq \Gamma(1)$, one has the genus formula (Theorem 3.1.1 of Diamond & Shurman)

$$g(X(\Gamma)) = 1 + \frac{\deg(X(\Gamma) \to X(1))}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{3} - \frac{\epsilon_\infty}{2}.$$

Here we have

1. $\epsilon_2 :=$ the number of elliptic points on $X(\Gamma)$ of period 2;

2. $\epsilon_3 :=$ the number of elliptic points on $X(\Gamma)$ of period 3;

3. $\epsilon_\infty :=$ the number of cusps on $X(\Gamma)$.

Let $\ell \geq 5$ be prime. Prove the following genera formulas:

a. $g(X_0(\ell)) = \begin{cases} \lfloor \frac{\ell+1}{12} \rfloor - 1 & \text{if } \ell \equiv 1 \pmod{12}, \\ \lfloor \frac{\ell+1}{12} \rfloor & \text{otherwise.} \end{cases}$

b. $g(X_1(\ell)) = 1 + \frac{(\ell-1)(\ell-11)}{24}$.

c. $g(X(\ell)) = 1 + \frac{(\ell^2-1)(\ell-6)}{24}$.

(For part c., let us take for granted that the number of cusps for $\Gamma(N)$ is $\epsilon_\infty(\Gamma(N)) = (1/2)N^2 \prod_{p|N}(1-1/p^2)$ when $N > 2$.)

Problem 19 (The Riemann-Roch theorem and Weierstrass equations of elliptic curves)**.** Let $E/\mathbb{C}$ be an elliptic curve, namely a genus 1 smooth curve (or compact Riemann surface) with a distinguished point O. In particular, E has divisors of the form $n(O)$. We will show that there is a cubic equation that we can associate to E.[4]

a. Show that $l(n(O)) = n$ for all sufficiently large n (*Hint*: see Problem 17). In fact, for which n does this hold?

[2] This has an algebraic geometry equivalent as well.
[3] The canonical divisor is unique up to linear equivalence.
[4] This exercise also extends to elliptic curves over fields other than $\mathbb{C}$.

b. Conclude that $L((O))$ only consists of the constant functions.

c. Conclude that $L(2(O))$ has a function x not in $L((O))$.

d. Conclude that $L(3(O))$ has a function y not in $L(2(O))$.

e. Conclude that $1, x, y, x^2, xy, y^2, x^3$ are all in $L(6(O))$.

f. However, $l(6(O)) = 6$. Therefore, $1, x, y, x^2, xy, y^2, x^3$ are linearly dependent over $\mathbb{C}$.

g. Conclude that we have a relation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with each $a_i \in \mathbb{C}$.[5]

[5] We have not actually shown that our elliptic curve is isomorphic to the curve defined by this cubic equation, but this is a start.