AWS 2021: Modular Groups
Problem Set 1

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1 Definitions and Notations

1. Given a ring R, define $M_2(R)$ as the set of 2×2 matrices over R, i.e.,
$$M_2(R) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \right\}.$$

2. Given a commutative ring R (with identity), define $\mathrm{GL}_2(R)$ as the set of 2×2 *invertible* matrices over R, i.e.,
$$\mathrm{GL}_2(R) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) : ad - bc \in R^\times \right\}.$$
If R has an ordering (for example, $R := \mathbb{R}$), we also define $\mathrm{GL}_2^+(R)$ as the set of 2×2 invertible matrices over R with positive determinant.

3. Define $\mathrm{SL}_2(R)$ as the set of 2×2 matrices over R with determinant 1, i.e.,
$$\mathrm{SL}_2(R) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R) : ad - bc = 1 \right\}.$$
Note that $\mathrm{GL}_2(R)$ and $\mathrm{SL}_2(R)$ are groups and $\mathrm{SL}_2(R) \subset \mathrm{GL}_2(R)$.
We let I denote the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

4. Recall that $\mathcal{H}$ or $\mathbb{H}$ denotes the upper half plane in the plane $\mathbb{C}$ of complex numbers, i.e.,
$$\mathcal{H} := \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}.$$

5. There is an action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C} \cup \{\infty\}$. For $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathcal{H}$, this action[1] is given by[2]
$$\frac{a\tau + b}{c\tau + d}$$
and
$$\gamma\infty := \frac{a}{c}.$$
This action is often called a *linear fractional transformation.*

6. Given $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathcal{H}$, we let $j(\gamma, \tau) := c\tau + d$.

7. $D \subset \mathbb{C}$ denotes the unit disk, i.e., $D := \{z \in \mathbb{C} : |z| < 1\}$.

[1] More specifically, there is an action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ – most of the time, we will be concerned with real matrices, and in fact matrices $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
[2] Note if $c \neq 0$ then $-d/c$ gets mapped to ∞ and if $c = 0, \infty$ gets mapped to ∞.

2 Introductory Problems

Problem 1. Which points of $\mathbb{C} \cup \{\infty\}$ are fixed by the linear fractional transformations given by

1. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$?

2. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$?

3. $\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$?

Problem 2. Show that if $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then the linear fractional transformation induced by γ maps $\mathbb{Q} \cup \{\infty\}$ to $\mathbb{Q} \cup \{\infty\}$.

Problem 3. Let R be a ring and let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$.

1. Given a matrix $\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \in \mathrm{GL}_2(R)$, evaluate the conjugate

$$\begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}^{-1}.$$

2. Given a matrix $\begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \in \mathrm{GL}_2(R)$, evaluate the conjugate

$$\begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix}^{-1}.$$

Problem 4 (PROMYS Summer 2014, *Geometry and Symmetry*, P2)**.** What is the stabilizer of $i \in \mathcal{H}$ under the action of $\mathrm{SL}_2(\mathbb{R})$? In other words, which elements of $\mathrm{SL}_2(\mathbb{R})$ fix i?

Problem 5 (PROMYS Summer 2014, *Geometry and Symmetry*, P1)**.** Let $B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ be the complex linear fractional transformation $Bz = \dfrac{z-i}{z+i}$. It turns out that B maps $\mathcal{H}$ into D, the unit disk, cf. Problem 11.

1. What is $B \cdot 0$?

2. What is $B \cdot i\infty$, i.e., $\lim_{y \to \infty} B \cdot iy$?

3. Show that B is a bijection by finding an inverse for B.

4. Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Show that $BSB^{-1} : D \to D$ is a rotation by π around the origin.

Problem 6 (Diamond & Shurman, Exercise 1.1.2)**.**

1. Show that $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)/|c\tau + d|^2$ for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.[3]

2. Show that $d\gamma(\tau)/d\tau = 1/(c\tau + d)^2$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

[3] You will need to use that $\det \gamma = 1$.

Problem 7. Given $\gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathcal{H}$, show that[4]

$$\gamma \begin{bmatrix} z \\ 1 \end{bmatrix} = j(\gamma, z) \begin{bmatrix} \gamma z \\ 1 \end{bmatrix}.$$

Problem 8 (Constructing elements of $\mathrm{SL}_2(\mathbb{Z})$)**.**

a. Find two integers $x, y \in \mathbb{Z}$ for which

$$\begin{bmatrix} 7 & x \\ 12 & y \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

How many more can you find?

b. Given two integers $a, b \in \mathbb{Z}$ not both zero, and their greatest common divisor

$$d := \gcd(a, b),$$

determine all pairs of integers $(x, y) \in \mathbb{Z}^2$ for which

$$\begin{bmatrix} a & x \\ b & y \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

(*Hint:* Use the Euclidean algorithm.)

3 Intermediate Problems

Problem 9. Show that the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathcal{H} \cup \mathbb{R} \cup \{\infty\}$ is indeed a group action, i.e.,

1. $I\tau = \tau$ for all $\tau \in \mathcal{H} \cup \{\infty\}$ and
2. $\gamma_1(\gamma_2\tau) = (\gamma_1\gamma_2)\tau$ for all $\gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}$.

Problem 10 (PROMYS Summer 2014, *Geometry and Symmetry*, P2)**.**

1. Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathcal{H}$ is transitive. In other words, show that for all $\tau_1, \tau_2 \in \mathcal{H}$, there is some $\gamma \in \mathrm{SL}_2(\mathbb{R})$ such that $\gamma\tau_1 = \tau_2$.
2. Show that the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$ is not transitive.

Problem 11. Let $B = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ be the complex linear fractional transformation $Bz = \frac{z-i}{z+i}$, just as in Problem 5. Show that if $z \in \mathcal{H}$ then $|Bz| < 1$, and so B maps $\mathcal{H}$ into the unit disk.

Problem 12. Given $\gamma_1, \gamma_2 \in \mathrm{GL}_2(\mathbb{R})$ and $\tau \in \mathcal{H}$, show that[5]

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2\tau)j(\gamma_2, \tau).$$

Problem 13 (Generators of $\mathrm{SL}_2(\mathbb{Z})$)**.** This exercise will show that $\mathrm{SL}_2(\mathbb{Z})$ is generated by "translation"

$$T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

together with "negative inversion"

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

i.e.,

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, T \rangle.$$

[4] Here, $\gamma \begin{bmatrix} z \\ 1 \end{bmatrix}$ is a multiplication of two matrices, γz is defined in Definition 5, and $j(\gamma, z) \begin{bmatrix} \gamma z \\ 1 \end{bmatrix}$ is a scalar multiple of a column matrix.

[5] You can show this by a direct computation, but you can also do so by computing $\gamma_1\gamma_2 \begin{bmatrix} \tau \\ 1 \end{bmatrix}$ in two different ways, cf. Problem 7.

a. Show that $S^2 = -I$, and that for $n \in \mathbb{Z}$ one has $T^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

b. Consider a matrix $\gamma := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ where $c \neq 0$.

   Suppose that $|a| \geq |c|$. Use the Euclidean algorithm to find an integer $q \in \mathbb{Z}$ for which

$$T^{-q}\gamma = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

   with $|a'| < |c|$.

c. Continuing the above, show that

$$ST^{-q}\gamma = \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix}$$

   with $|c| > |c''|$.

d. Continuing this process of applying ST^k to γ for various $k \in \mathbb{Z}$ a finite number of times, we may assume that γ is upper triangular,

$$\gamma = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}.$$

   Determine what a and d are, and conclude that $\gamma \in \langle S, T \rangle$.

4 Advanced Problems

Problem 14 (Ping-Pong Lemma)**.** Let G be a group generated by two elements a and b. Suppose G acts on a set X and we can find two subsets $X_1, X_2 \subset X$ such that $X_1 \not\subset X_2$ and $X_2 \not\subset X_1$, and such that for every integer $n \neq 0$ we have

$$a^n(X_1) \subset X_2, \qquad b^n(X_2) \subset X_1.$$

Show that G is freely generated by a and b. (*Hint:* Write any $g \in G$ as a word in a and b and see where it sends X_1 or X_2.)

Problem 15. Use the previous problem to show that the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

is free. To do this, let $\mathrm{SL}_2(\mathbb{Z})$ act on column vectors in $\mathbb{R}^2$ by usual matrix multiplication, and let

$$X_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| < |y| \right\},$$

and

$$X_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : |x| > |y| \right\}.$$

Problem 16. Show that the group constructed in the previous problem is free by instead letting it act on the upper half plane, with

$$X_1 = \{x + iy \in \mathcal{H} : |x| < 1\},$$

and

$$X_2 = \{x + iy \in \mathcal{H} : |x| > 1\}.$$