EXERCISES IN NONARCHIMEDEAN GEOMETRY: AWS 2020

EVAN WARNER

For each exercise, you may assume all results and exercises stated up until that point – sometimes we will admit certain facts as black boxes.

Throughout, let K be a field equipped with a nonarchimedean (rank one) absolute value $|\cdot| : K \to \mathbf{R}_{\geq 0}$. Let

$$R = \{t \in K : |t| \leq 1\}$$

be its valuation ring,

$$\mathfrak{m} = \{t \in K : |t| < 1\}$$

the maximal ideal of R, and $k = R/\mathfrak{m}$ the residue field.

1. Affinoid algebras

Exercise 1.1. Let a *ball* in K be a subset of the form $\{t \in K : |t - a| < r\}$ or $\{t \in K : |t - a| \leq r\}$ with $a \in K$ (the center) and $r \in \mathbf{R}_{>0}$ (the radius). For example, R and $\mathfrak{m}$ are balls.
a) Show that any point of a ball in K is a center of that ball.
b) Show that if any two balls in K intersect, then one is contained in the other.
c) Show that every ball in K is both open and closed in the metric topology induced by the absolute value.
d) Show that K is totally disconnected in this topology.

Exercise 1.2 (Serre)**.** If V is an open set in the product topology on K^n, say that a function $V \to K$ is *locally analytic* if it can be locally written as a convergent power series with coefficients in K. Define the category of *naïve K-analytic manifolds* to be the category of locally ringed spaces that are locally isomorphic to a pair $(V, \mathcal{O}_V^{\text{l-an}})$ where $\mathcal{O}_V^{\text{l-an}}$ is the sheaf of locally analytic functions on V. There is an obvious notion of dimension. Suppose that K is a local field and let q be the cardinality of the residue field. Show that any n-dimensional compact naïve K-analytic manifold is isomorphic to a disjoint union of s copies of the unit polydisc R^n, where $s < q$.

Remark 1.3. In the four-page article [5], Serre shows furthermore that s as above is uniquely determined. It is sometimes called the *Serre invariant*.

Clearly we need to do something else to get a "reasonable" class of analytic spaces over a nonarchimedean field. For $n \geq 1$, define the n-dimensional *Tate algebra* T_n over K, also denoted by $K\langle X_1, \ldots, X_n\rangle$, to be the subring of $K[[X_1, \ldots, X_n]]$ consisting of power series that converge on R^n. We will often denote the n-tuple $X_1, \ldots, X_n$ simply as X and write X^J for a multi-index $J = \{j_1, \ldots, j_n\}$ to mean $\prod X_i^{j_i}$.

Exercise 1.4. Check that

$$T_n = \left\{\sum c_J X^J : c_J \in K, |c_J| \to 0 \text{ as } ||J|| \to \infty\right\},$$

where $||J|| = \sum j_i$.

The *Gauss norm* on T_n is defined by

$$\left|\left|\sum c_J X^J\right|\right| = \max_J |c_J|.$$

Exercise 1.5 (Taken from [4])**.**
a) Check that $||f + g|| \leq \max(||f||, ||g||)$, $||cf|| = |c| \cdot ||f||$, and $||fg|| \leq ||f|| \cdot ||g||$ for all $f, g \in T_n$ and $c \in K$.
b) Check that T_n is complete for the metric induced by the Gauss norm.
c) Check that actually $||fg|| = ||f|| \cdot ||g||$ for all $f, g \in T_n$ by first scaling so that f and g both have unit norm and then reducing modulo $\mathfrak{m}$.

d) Let $\overline{K}$ be an algebraic closure of K endowed with the unique absolute value extending the given one on K. Again by using the scaling trick and reducing modulo $\mathfrak{m}$, show that

$$||f|| = \sup_x |f(x)| = \max_x |f(x)|$$

where $x = (x_1, \dots, x_n)$ varies over all elements of $\overline{K}^n$ such that $|x_j| \leq 1$.
e) Give an example of a $f \in \mathbf{Q}_p\langle X \rangle$ such that $||f|| > \sup_{x \in \mathbf{Z}_p} |f(x)|$.

This suggests that, as in algebraic geometry, one should in some way include $\overline{K}$-valued points in the underlying set of a K-analytic space. Using techniques similar to those used in complex analytic geometry (i.e., the Weierstrass preparation theorem), one can show that the n-dimensional Tate algebra is noetherian, Jacobson (every prime ideal is the intersection of the maximal ideals containing it), regular, and a unique factorization domain. Furthermore every ideal is closed with respect to the Gauss norm and reduction modulo every maximal ideal yields a field that has finite degree over K.

Exercise 1.6. Using these facts, show that the set $\mathrm{MaxSpec}(T_n)$ of maximal ideals of the n-dimensional Tate algebra is naturally in bijection with $\overline{R}^n$ modulo the Galois action, where $\overline{R} = \{t \in \overline{K} : |t| \leq 1\}$ is the valuation ring of $\overline{K}$.

In particular if for $f \in T_n$ and $x = \mathfrak{m}_x \in \mathrm{MaxSpec}(T_n)$ we let $f(x)$ denote the image of f in $T_n/\mathfrak{m}_x$, we get the more intrinsic formula

$$||f|| = \sup_x |f(x)| = \max_x |f(x)|$$

where now x runs over $\mathrm{MaxSpec}(T_n)$.

A K*-affinoid algebra* A is a K-algebra that can be written as a quotient of some T_n. If A is a K-affinoid algebra, let $\mathrm{Sp}(A)$ denote the set $\mathrm{MaxSpec}(A)$. All affinoid algebras are noetherian and Jacobson, and all residue fields $A/\mathfrak{m}$ with $\mathfrak{m} \in \mathrm{Sp}(A)$ are finite over K.

Exercise 1.7. Check that an element of an affinoid algebra A is nilpotent if and only if it lies in every maximal ideal of A and explain the geometric significance of this fact.

Exercise 1.8. Show that the assignment $A \mapsto \mathrm{Sp}(A)$ is contravariantly functorial. If $A \simeq T_n/I$, describe the set $\mathrm{Sp}(A)$.

If $A \simeq T_n/I$, we can equip A with the structure of a Banach K-algebra by taking the Gauss norm on T_n and then taking the quotient norm. This quotient norm may depend on the presentation T_n/I, but it is a fact from nonarchimedean analysis that the resulting topology does not. Furthermore all K-algebra maps of K-affinoid algebras are automatically continuous with respect to this topology.

Exercise 1.9. If A is a K-affinoid algebra, then there is a canonical *semi-norm*: for $f \in A$, let

$$|f|_{\mathrm{sup}} = \sup_{x \in \mathrm{Sp}(A)} |f(x)|.$$

By previous work, we know that this coincides with the Gauss norm if $A = T_n$.
a) Show that $|\cdot|_{\mathrm{sup}}$ is a semi-norm (i.e., it is a norm except that it may not be the case that $|f|_{\mathrm{sup}} = 0$ implies $f = 0$) which is a norm if A is reduced.
b) Show that $|f^n|_{\mathrm{sup}} = |f|^n_{\mathrm{sup}}$.
c) If $\phi : A \to B$ is a morphism of K-affinoid algebras, then $|\phi(f)|_{\mathrm{sup}} \leq |f|_{\mathrm{sup}}$ for all $f \in A$.

One can show that in fact $|f|_{\mathrm{sup}} = \max_{x \in \mathrm{Sp}(A)} |f(x)|$; i.e., a maximum modulus principle holds.

Exercise 1.10 (Taken from [4])**.** Let A be a K-affinoid algebra and let A^0 denote the subset of power-bounded elements, where $a \in A$ is *power-bounded* if the sequence $\{a^n\}_{n \geq 1}$ is bounded.
a) Show that A^0 is an R-algebra.
b) Prove that the assignment $A \mapsto A^0$ is functorial in the category of K-affinoid algebras.
c) Show that the map $\mathrm{Hom}(T_n, A) \to (A^0)^n$ defined by $\phi \mapsto (\phi(X_1), \dots, \phi(X_n))$ is a bijection. Compare this to the universal mapping property of polynomial algebras in algebraic geometry.

Exercise 1.11. If A is a K-affinoid algebra, define a *Zariski topology* on $\mathrm{Sp}(A)$. State and prove a Nullstellensatz in this context.

Exercise 1.12. If A is a K-affinoid algebra, define a *canonical topology* on $\mathrm{Sp}(A)$ by taking the topology generated by sets of the type

$$\{x \in \mathrm{Sp}(A) : |f(x)| \leq \epsilon\}$$

for $f \in A$ and $\epsilon \in \mathbf{R}_{>0}$. Show that the canonical topology is Hausdorff, totally disconnected, and functorial in A.

2. Affinoid subdomains

To construct the "correct" category of K-analytic spaces, we need a topology intermediate in strength between the Zariski and canonical topologies. Unfortunately no ordinary topology will do the job[1] so we instead define a (rather mild) Grothendieck topology. If A is a K-affinoid algebra, a subset $U \subseteq \mathrm{Sp}(A)$ is called an *affinoid subdomain* if there exists a map $A \to B$ of K-affinoid algebras such that the induced map $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ lands in U and is universal for such maps in the sense that for any other map $A \to C$ of K-affinoid algebras, there is a unique map $B \to C$ making the obvious diagram commute if and only if $\mathrm{Sp}(C)$ lands in U. By the usual category theory, such a B is unique up to unique isomorphism; call it A_U.

Exercise 2.1. Prove that the natural map $\mathrm{Sp}(A_U) \to \mathrm{Sp}(A)$ is an injection onto U. Prove that if $V \subseteq U \subseteq \mathrm{Sp}(A)$ with U an affinoid subdomain of $\mathrm{Sp}(A)$, then V is an affinoid subdomain of U if and only if V is an affinoid subdomain of $\mathrm{Sp}(A)$. Prove that if $V \subseteq U$ is an inclusion of affinoid subdomains, then there is an induced map $A_U \to A_V$ that is transitive with respect to compositions of inclusions.

Exercise 2.2. Show that the category of K-affinoid algebras has pushouts by constructing a "completed tensor product" as follows: first, if A and B are K-affinoid algebras, choose presentations $A \simeq T_m/I$ and $B \simeq T_n/J$ and let $A \widehat{\otimes}_K B \simeq T_{m+n}/(I' + J')$, where I' and J' are the ideals generated by I and J via the obvious projection maps. Then boostrap to the case of a general pushout $A \widehat{\otimes}_C B$.

Exercise 2.3. Prove that if $U, U' \subseteq \mathrm{Sp}(A)$ are affinoid subdomains, then $U \cap U'$ is also an affinoid subdomain with

$$A_{U \cap U'} \simeq A_U \widehat{\otimes}_A A_{U'}.$$

Exercise 2.4. Show that the inverse image of an affinoid subdomain along the map $\mathrm{Sp}(B) \to \mathrm{Sp}(A)$ induced from a map of K-affinoid algebras $A \to B$ is itself an affinoid subdomain.

The extremely convenient fact of rigid geometry is that there exist a plentiful supply of affinoid subdomains; in other words, there are many subsets of $\mathrm{Sp}(A)$ that are themselves spectra of K-affinoid algebra. This is analogous to the fact in algebraic geometry that a localization of an algebra of finite type over a ring with respect to finitely many functions is still an algebra of finite type (thanks to the isomorphism $R[f^{-1}] \simeq R[t]/(ft - 1)$). Here the algebraic operation will not be localization, but rather some sort of completed localization.

If A is a K-affinoid algebra, define the *relative Tate algebra* in n variables $A\langle X \rangle = A\langle X_1, \dots, X_n \rangle$ exactly as one defined T_n, except replacing K by A.

Exercise 2.5. Prove that relative Tate algebras are still K-affinoid. State and prove a universal mapping property of relative Tate algebras, and use it to conclude that $(A\langle X \rangle)\langle Y \rangle \simeq A\langle X, Y \rangle$.

Given a K-affinoid algebra A and a collection of elements $f_1, \dots, f_n, g \in A$ with no common zeroes, define

$$A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle := A\langle X_1, \dots, X_n \rangle/(gX_1 - f_1, \dots gX_n - f_n).$$

Exercise 2.6. With notation as above, let $U = \{x \in \mathrm{Sp}(A) : |f_i(x)| \leq |g(x)| \text{ for all } i\}$. Show that U is an affinoid subdomain with

$$A_U \simeq A\left\langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \right\rangle.$$

Affinoid subdomains of the form described in the above example are called *rational subdomains.*

Exercise 2.7. Show that rational subdomains are open in the canonical topology, and even form a basis thereof. Show that pullbacks of rational subdomains are rational (and describe them concretely).

[1]At least if we insist that the underlying set is $\mathrm{Sp}(A)$. The theories of Berkovich spaces and adic spaces do involve actual topologies, at the expense of adding more underlying points.

It can be shown that all affinoid subdomains are open in the canonical topology, and even that every affinoid subdomain is a finite union of rational domains (Gerritzen-Grauert theorem).

Exercise 2.8. In contrast to algebraic geometry, if $U \subseteq \mathrm{Sp}(A)$ is an affinoid subdomain then there are in general many Zariski-closed subspaces of U that do not extend to Zariski-closed subspaces of $\mathrm{Sp}(A)$. This is exactly analogous to the situation in complex-analytic geometry, where analytic functions have domains of holomorphy that may be much smaller than the whole space. Explain why this is equivalent to the existence of a non-maximal prime ideal $\mathfrak{q} \subset A_U$ such that $\mathfrak{p} := \mathfrak{q} \cap A$ does not satisfy $\mathfrak{p}A_U = \mathfrak{q}$. Give an explicit example of this (this is somewhat tricky; see pp. 61-63 of [1] if you get stuck). Explain why in the definition of $\mathrm{Sp}(A)$ we considered only maximal ideals rather than all prime ideals.

3. Rigid spaces

By Exercise 2.7, there is no ordinary topology strictly coarser than the canonical topology but where rational subdomains are open sets. We rather pass to a Grothendieck topology, where we are allowed to restrict what covers are allowed. Call $U \subseteq \mathrm{Sp}(A)$ an *admissible open* if it is covered by affinoid subdomains $\{U_i\}$ such that for each affinoid subdomain $V \subseteq U$, there are finitely many elements of $\{U_i\}$ that cover V. Call a collection of admissible opens $\{U_i\}$ of $\mathrm{Sp}(A)$ an *admissible cover* of its union U if for every affinoid subdomain V of U the restriction of the collection to V has a refinement by a covering consisting of finitely many affinoid subdomains.

Exercise 3.1. As a sanity check, show:
a) The union of an admissible cover is an admissible open.
b) The covering in the definition of an admissible open is an admissible cover.
c) A finite union of affinoids is admissible, with said finite union giving an admissible cover.

Exercise 3.2. Inside $\mathrm{Sp}(T_1)$, let (informally) $U = \{|t| < 1\}$ and $V = \{|t| = 1\}$. Clearly V is an affinoid subdomain. Using the maximum modulus principle, show that U is an admissible open and $\{U, V\}$ is *not* an admissible cover of $\mathrm{Sp}(T_1)$.

Exercise 3.3. More generally, show that any Zariski open is an admissible open. Show that any Zariski open cover is an admissible cover.

Given a K-affinoid algebra A, define the *Tate topology* on $\mathrm{Sp}(A)$ to have objects the admissible open subsets and coverings the admissible open coverings.

Exercise 3.4. Check that this yields a Grothendieck topology.

Tate's Acyclicity Theorem states that the assignment $U \mapsto A_U$ uniquely extends to a sheaf, the *structure sheaf* $\mathcal{O}_A$, on the Tate topology of $\mathrm{Sp}(A)$ (and further that this sheaf has no higher cohomology). The proof is not easy and proceeds by successive reduction to simpler cases until a direct computation can be made. With this in hand, one can formally define the notion of a "locally ringed G-topologized space" and define an *K-affinoid space* to be the locally ringed G-topologized space $(\mathrm{Sp}(A), \mathcal{O}_A)$. For details, see Chapter 5 of [1] or Chapter 9 of [2]. We will commit the notational sin of referring to $(\mathrm{Sp}(A), \mathcal{O}_A)$ simply as $\mathrm{Sp}(A)$.

Exercise 3.5. Show that the assignment $A \mapsto \mathrm{Sp}(A)$ is a *fully faithful* contravariant functor from K-affinoid algebras to locally K-ringed G-topologized spaces.

A *rigid-analytic space* over K is then a locally ringed G-topologized space $(X, \mathcal{O}_X)$ that is locally isomorphic to K-affinoid spaces. A morphism of rigid-analytic spaces is just a morphism of locally K-ringed G-topologized spaces.

Exercise 3.6. Sanity check: show that an admissible open subset of a rigid-analytic space is a rigid-analytic space.

Exercise 3.7. Let X be a rigid-analytic space over K and Y a K-affinoid space. Using Exercise 3.5, show that the natural map

$$\mathrm{Mor}(X, Y) \to \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

is a bijection.

Exercise 3.8 (Taken from [4]). A rigid-analytic space X is *connected* if there does not exist an admissible open covering $\{U, V\}$ of X with U, V nonempty and $U \cap V = \emptyset$.
a) Prove that $\mathrm{Sp}(A)$ is connected if and only if A has no nontrivial idempotents. In particular, using the Tate topology we recover a "reasonable" notion of connectedness.
b) Carefully define a good notion of "connected component" of a rigid space X, and prove that the connected components form an admissible cover of X.
c) Prove that a rigid space X is connected if and only if $\mathcal{O}_X(X)$ has no nontrivial idempotents.

Exercise 3.9. Pick a $c \in K$ with $0 < |c| < 1$. Let $\{D_j\}_{j \geq 1}$ be copies of the unit disc $\mathrm{Sp}(T_n)$, with coordinates $x_{1,j}, \ldots, x_{n,j}$. Define maps $D_j \to D_{j+1}$ by sending $x_{n,j+1} \mapsto cx_{n,j}$; i.e., D_j is the affinoid subdomain $\{|x_{i,j}| \leq |c| \text{ for all } i\}$ in D_{j+1}.
a) Carefully define a rigid space by gluing the D_j along these maps. It is called *affine n-space* and is denoted by $\mathbf{A}_K^{n,\mathrm{an}}$.
b) Show that, as a set, $\mathbf{A}_K^{n,\mathrm{an}}$ coincides with the set of closed points of the scheme $\mathbf{A}_K^n$.
c) Prove the following universal property: for any rigid-analytic space X over K, the natural map of sets

$$\mathrm{Mor}(X, \mathbf{A}_K^{n,\mathrm{an}}) \to \mathcal{O}_X(X)^n$$

is a bijection. In particular this shows that the construction of $\mathbf{A}_K^{n,\mathrm{an}}$ is independent of the choice of c.

Let $\mathcal{X}$ be a scheme locally of finite type over K. An *analytification* of $\mathcal{X}$ is a rigid-analytic space $\mathcal{X}^{\mathrm{an}}$ together with a morphism of localy K-ringed G-topologized spaces $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ that is universal for maps from rigid-analytic spaces over K. By general nonsense this specifies $\mathcal{X}^{\mathrm{an}}$ up to unique isomorphism.

Exercise 3.10. Verify that $\mathbf{A}_K^{n,\mathrm{an}}$ is an analytification. Construct analytifications of arbitrary affine K-schemes of finite type, then of arbitrary schemes locally of finite type over K. Show that the underlying map of sets $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ identifies the points of $\mathcal{X}^{\mathrm{an}}$ with the closed points of $\mathcal{X}$. Define what the analytification of a morphism of schemes locally of finite type over K should be and construct it.

Exercise 3.11. Verify that the category of rigid-analytic spaces over K admits fiber products and that analytification respects fiber products in the obvious sense.

Just as in complex-analytic geometry, there is a GAGA principle in rigid geometry: in particular, rigid-analytic morphisms of analytifications of proper schemes over K must be algebraic.

Exercise 3.12 (Open-ended). Take familiar properties of schemes and morphisms of schemes and try to generalize them to rigid-analytic spaces and prove their basic properties. For example: quasi-compact, closed immersion, separated, quasi-separated, of dimension n, finite, étale, smooth, unramified, flat. Properness is tricky!

4. Formal geometry

Rigid spaces can also be constructed as "generic fibers" of certain formal schemes. Define $R\langle X\rangle = R\langle X_1, \ldots, X_n\rangle$ to be the subalgebra of $R[[X_1, \ldots, X_n]]$ consisting of power series with coefficients in R tending to 0. A *topologically finitely presented* (or tfp) R-algebra is an R-algebra isomorphic to $R\langle X\rangle/I$, where I is a finitely generated ideal. A tfp R-algebra is *admissible* if it is flat over R, which is equivalent to R not having any $\mathfrak{m}$-torsion. Warning: tfp R-algebras are in general nonnoetherian (unless R is a dvr), so one sometimes needs serious algebraic input in order to work with them.

Exercise 4.1. Let $\pi \in R$ be such that $0 < |\pi| < 1$; such an element is called a *pseudouniformizer*. Show that $R\langle X\rangle$ is equal to the π-adic completion of $R[X]$. If R is not discretely valued, show that this is *not* the same as the $\mathfrak{m}$-adic completion!

Exercise 4.2. Associated to a tfp R-algebra $\mathscr{A}$, we can consider the "special fiber" or "reduction" $\mathscr{A} \otimes_R k$ and the "generic fiber" $\mathscr{A} \otimes_R K = \mathscr{A}\left[\frac{1}{\pi}\right]$. Show that the special fiber is a k-algebra of finite type and that the generic fiber is an affinoid K-algebra.

If $\mathscr{A}$ is a tfp R-algebra and $f \in \mathscr{A}$, set

$$\mathscr{A}\langle f^{-1}\rangle = \varprojlim_n \left(\mathscr{A}[f^{-1}]/\pi^n\right)$$

where π is a pseudouniformizer of R.

Exercise 4.3. With notation as above, prove that the natural map $\mathscr{A}\langle X_1\rangle/(1-fX_1) \to \mathscr{A}\langle f^{-1}\rangle$ sending X_1 to f^{-1} is an isomorphism. In particular $\mathscr{A}\langle f^{-1}\rangle$ is independent of the choice of π.

Exercise 4.4. Let $\mathscr{A}$ be a tfp R-algebra and let $\overline{X} = \mathrm{Spec}(\mathscr{A}/\mathfrak{m}\mathscr{A})$. For $f \in \mathscr{A}$, let $\overline{X}_f$ denote the nonvanishing locus of f in $\overline{X}$. Define a presheaf on subsets of $\overline{X}$ of this kind via the assignment $\overline{X}_f \mapsto \mathscr{A}\langle f^{-1}\rangle$. Show that this presheaf satisfies the sheaf axioms and deduce that this presheaf extends to a sheaf $\mathcal{O}_{\mathscr{A}}$ on $\overline{X}$.

We let $\mathrm{Spf}(\mathscr{A})$ denote the locally ringed space $(\overline{X}, \mathcal{O}_{\mathscr{A}})$ as above and call such spaces *affine tfp formal schemes*. A *tfp formal scheme* $(\mathscr{X}, \mathcal{O}_{\mathscr{X}})$ over R is a quasi-compact locally R-ringed space that is locally isomorphic to affine tfp formal schemes. An *admissible formal scheme* over R is a tfp formal scheme over R locally isomorphic to affine tfp formal schemes $\mathrm{Spf}(\mathscr{A})$ where $\mathscr{A}$ is an admissible R-algebra.

Exercise 4.5. Check that we can extend Exercise 4.2 to construct a special fiber (or reduction) functor $\mathscr{X} \mapsto \mathscr{X}_k$ from tfp formal schemes over R to schemes of finite type over k, and a generic fiber functor $\mathscr{X} \mapsto \mathscr{X}_K$ from tfp formal schemes over R to rigid spaces over K.

A potential source of confusion is the following. Let $\mathcal{X}$ be an R-scheme of finite type. One can construct the generic fiber $\mathcal{X}_K$ in the usual scheme sense (i.e., $\mathcal{X}_K = \mathcal{X} \otimes_R K$) and then take the analytification $\mathcal{X}_K^{\mathrm{an}}$, *or* one can form the π-adic completion $\mathscr{X}$ of $\mathcal{X}$ and then take the generic fiber $\mathscr{X}_K$ in the sense of the preceding exercise.

Exercise 4.6. With notation as above:
a) If $\mathcal{X} = \mathrm{Spec}(R[t])$ compute $\mathcal{X}_K^{\mathrm{an}}$ and $\mathscr{X}_K$, and note that they are different.
b) Show that nevertheless for any $\mathcal{X}$ there is a canonical map of rigid spaces $\iota_{\mathcal{X}} : \mathcal{X}_K^{\mathrm{an}} \to \mathscr{X}_K$ and identify it in the above case.
c) If $\mathcal{X} = \mathbf{P}_R^n$ is projective space, show that $\iota_{\mathcal{X}}$ is an isomorphism.

More generally, $\iota_{\mathcal{X}}$ is always an isomorphism whenever $\mathcal{X}$ is proper over R.

Given a rigid space X over K, an admissible formal R-scheme $\mathscr{X}$ together with an isomorphism $\mathscr{X}_K \simeq X$ is called a *formal model* of X. Raynaud proved that every quasi-paracompact[2] quasi-separated rigid space X over K has a formal model, and in fact if $\mathscr{X}_1$ and $\mathscr{X}_2$ are two such then there is a third $\mathscr{X}_3$ and maps $\mathscr{X}_3 \to \mathscr{X}_1$ and $\mathscr{X}_3 \to \mathscr{X}_2$ that are both blowups of closed subschemes of the special fibers ("admissble formal blowups"). Furthermore any morphism of rigid schemes has a formal model. The power of the theory comes from the fact that often one can choose the formal models to inherit the properties of the rigid space or morphism of rigid spaces under consideration; for example, any quasi-compact open immersion comes from a (Zariski-)open immersion into a suitable formal model.

Exercise 4.7. Let $\mathscr{X}$ be a formal model of X. Given a point x of X, we can consider $x \in \mathrm{Sp}(A)$ where $A = \mathscr{A} \otimes_R K$ and $\mathscr{A}$ is some affine open of $\mathscr{X}$. Viewing x as a maximal ideal $\mathfrak{m}_x$, the morphism $A \to A/\mathfrak{m}_x$ takes $\mathscr{A}$ into the valuation ring of $A/\mathfrak{m}_k$. Reducing mod π, we get a map from $\mathscr{A} \otimes_R k$ onto a finite extension of k; i.e., a point of $\mathrm{Spec}(\mathscr{A} \otimes_R k)$, hence of $\mathscr{X}_k$. Show that this point is independent of the choices made, so yields a *specialization map* $\mathrm{sp} : X \to \mathscr{X}_k$. Show that the preimage of a Zariski-open set of $\mathscr{X}_k$ is a finite union of rational subsets of X (so in particular, it is an admissible subset).

The preimage of a point x of the special fiber under sp is sometimes called the *tube* of x and denoted by $]x[$.

5. Curves

A single rigid space has many formal models, hence many different reductions. For the purposes of algebraic or arithmetic geometry, some may be better than others.

Exercise 5.1. Let $D = \mathrm{Sp}(T_1)$ be the unit disk. There is an "obvious" formal model $\mathrm{Spf}(R\langle X\rangle)$ of D whose special fiber is $\mathbf{A}_k^1$.
a) Let π be a pseudouniformizer of R and consider the formal scheme $\mathscr{D}$ obtained by glueing $R\langle X/\pi\rangle$ and $R\langle \pi/X, X\rangle = R\langle X, Y\rangle/(XY - \pi)$ along $R\langle X/\pi, \pi/X\rangle$. Show that the generic fiber of $\mathscr{D}$ is D, obtained by glueing the sets

[2] I.e., there exists an admissible open covering $\{X_i\}_{i\in I}$ by quasi-compact rigid spaces X_i such that for each index $i \in I$ there are only finitely many $j \in I$ such that $X_i \cap X_j$ is nonempty.

$\{t \in K : |t| \leq |\pi|\}$ and $\{t \in K : |t| \geq |\pi|\}$ along $\{t \in K : |t| = |\pi|\}$. Show that the special fiber of $\mathscr{D}$ consists of two components, an affine line and a projective line intersecting at an ordinary double point. What is the tube of the singularity?
b) Generalize by splitting up D into more concentric annuli. What sort of reductions can you get?

Exercise 5.2. Let $X = \mathbf{P}_K^{1,\mathrm{an}}$. Again there is an "obvious" formal model given by the formal projective line over R, with reduction $\mathbf{P}_k^1$. Find a formal model of X whose special fiber consists of two copies of $\mathbf{P}_k^1$ intersecting at an ordinary double point by splitting up $\mathbf{P}_K^{1,\mathrm{an}}$ into two balls and an annulus in between.

As should be fairly clear from the last two exercises, one can often think of a formal model for a rigid space in a somewhat concrete way via a particular choice of admissible affinoid covering. Recall that if A is a *reduced* K-affinoid algebra, its sup-norm is a norm. Since the sup-norm is canonical, we can define a *canonical reduction*

$$\tilde{A} := \mathring{A}/\check{A}.$$

where

$$\mathring{A} := \{f \in A : |f|_{\mathrm{sup}} \leq 1\}$$

and

$$\check{A} := \{f \in A : |f|_{\mathrm{sup}} < 1\}.$$

Unfortunately, it is not always the case that $\mathring{A}$ is even of topologically finite type over R.

Exercise 5.3. Check that nonetheless $\tilde{A}$ is a reduced k-algebra of finite type, there is always a *canonical specialization map* $\mathrm{Csp} : \mathrm{Sp}(A) \to \mathrm{Spec}(\tilde{A})$, and the preimage of a Zariski open set under Csp is admissible open.

If K is discretely valued or algebraically closed (or more generally, if K is *stable*; see 3.6 of [2] for the definition) and A satisfies $|A|_{\mathrm{sup}} = |K|$, then it is known that $\mathring{A}$ is topologically of finite type over R (see 6.4 of [2] for this and many related results).

Exercise 5.4. Assume that K is discretely valued or algebraically closed and that X is a reduced rigid-analytic space over K. Suppose you are given an admissible affinoid covering $\{X_i = \mathrm{Sp}(A_i)\}_{i \in I}$ of X such that:

- We have $|A_i|_{\mathrm{sup}} = |K|$ for each $i \in I$.
- If $X_i \cap X_j$ is nonempty, then there exists an open affine subset $U_{i,j}$ of the reduction $\mathrm{Spec}(\tilde{A}_i)$ such that $X_i \cap X_j$ is the inverse image of $U_{i,j}$ under the canonical specialization map.

From this data, construct a "formal model" that is topologically of finite type (but not necessarily admissble) $\mathscr{X}$ of X (and hence a reduction $\mathscr{X}_k$).

A *rigid curve* is a rigid space of dimension 1. We have not developed the dimension theory of rigid spaces, but due to the Noether normalization theorem in rigid geometry this is equivalent to being covered by affinoid spaces of the form $\mathrm{Sp}(A)$, where A is an affinoid algebra that can be written as a module-finite algebra over T_1. Using similar reasoning to the complex case, one can show that every proper rigid curve is in fact projective algebraic, so there is not much loss of generality in just considering open subspaces of analytifications of projective curves over K.

Exercise 5.5. Let $\phi : \mathscr{X} \to \mathscr{Y}$ be a morphism of admissible formal schemes over R. Let $x \in \mathscr{X}_k$ and let $y = \phi_k(x) \in \mathscr{Y}_k$ be its image under the morphism of special fibers induced by ϕ. Suppose that ϕ_k is étale at x. Prove that the induced morphism of the corresponding tubes $]x[\to]y[$ is an isomorphism. (Hint: reduce to using a standard étale map and lift the polynomial you get.)

Exercise 5.6. Let X be any rigid curve and $\mathcal{X}$ a formal model of X.
a) Show that x is a smooth point of the reduction $\mathcal{X}_k$ if and only if the tube $]x[$ is isomorphic to $\{t \in K : |t| < 1\}$.
b) Show that x is an ordinary double point if and only if $]x[$ is isomorphic to $\{t \in K : \pi < |t| < 1\}$ for some pseudouniformizer π.

In complex-analytic geometry, the uniformization theorem tells us that compact complex curves can be expressed as analytic quotients of certain simple spaces (the complex plane in genus 1 and the upper half plane or unit disk in genus > 1). The situation is not quite as nice in rigid geometry, but certain proper rigid curves do have reasonable uniformizations.

Exercise 5.7. Let $q \in K^*$ with $0 < |q| < 1$. We consider q as an automorphism of the rigid space $\mathbf{G}_{m,K}^{\mathrm{an}}$ sending $z \mapsto qz$.
a) Define the quotient $T = \mathbf{G}_{m,K}^{\mathrm{an}}/q^{\mathbf{Z}}$ as a locally ringed space together with a map $\pi : \mathbf{G}_{m,K}^{\mathrm{an}} \to T$.
b) Let $A(r_1, r_2) = \{t \in \mathbf{G}_{m,K}^{\mathrm{an}} : r_1 \leq |t| \leq r_2\}$. Using domains of the form $\pi(A(r_1, r_2))$ cover T by affinoids with affinoid intersections and thereby show that T is a connected rigid space.

One can show without difficulty that T is a smooth proper rigid curve, and hence is the analytification of some projective algebraic curve.

Exercise 5.8. Prove that T is the analytification of a curve of genus one by pure thought (no rigid-analytic computations!).

Our construction of T gives a natural base point, so we can consider T as the analytification of an elliptic curve. It is called the *Tate curve*, and its construction motivated the entire field of rigid analysis.

Exercise 5.9. Using the formal model associated to the affinoid cover of T you wrote down in Exercise 5.7 and formal GAGA, show that T is the analytification of a curve with split toric reduction.

In the complex-analytic case, one can explicitly write down a Weierstrass elliptic curve E_q whose analytification is $\mathbf{C}^*/q^{\mathbf{Z}}$. We can do the same thing here, and the relevant power series are actually given by the "same" formulas as in the complex case.

Exercise 5.10. Let $q \in K^*$ be such that $|q| < 1$. Define

$$s_k(q) = \sum_{n \geq 1} \frac{n^k q^n}{1 - q^n}, \quad a_4(q) = -5s_3(q), \quad a_6 = -\frac{5s_3(q) + 7s_5(q)}{12}$$

as elements of K.
a) Check that these series converge.
b) For $t \in \mathbf{G}_{m,K}^{\mathrm{an}}$, let

$$X_q(t) = \sum_{n \in \mathbf{Z}} \frac{q^n t}{(1 - q^n t)^2} - 2s_1(q), \qquad Y_q(t) = \sum_{n \in \mathbf{Z}} \frac{(q^n t)^2}{(1 - q^n t)^3} + s_1(q).$$

Show that $X_q(t)$ and $Y_q(t)$ converge when $t \notin q^{\mathbf{Z}}$.
c) Prove that $X_q(qt) = X_q(t) = X_q(t^{-1})$, $Y_q(qt) = Y_q(t)$, and $Y_q(t^{-1}) = -Y_q(t) - X_q(t)$.
d) Expand $X_q(t)$ and $Y_q(t)$ as power series in q. Namely, prove that if $|q| < |t| < |q^{-1}|$

$$X_q(t) = \frac{t}{(1-t)^2} + \sum_{d \geq 1} \left(\sum_{m \mid d} m(t^m + t^{-m} - 2) \right) q^d$$

and

$$Y_q(t) = \frac{t^2}{(1-t)^3} + \sum_{d \geq 1} \left(\sum_{m \mid d} \frac{(m-1)m}{2} t^m - \frac{m(m+1)}{2} t^{-m} + m \right) q^d.$$

e) Define E_q to be the elliptic curve over K given by the Weierstrass equation $y^2 + xy = x^3 + a_4(q)x + a_6(q)$. Construct a morphism $\phi : \mathbf{G}_{m,K}^{\mathrm{an}} \to E_q^{\mathrm{an}}$ by sending $t \mapsto (X_q(t), Y_q(t))$ if $t \notin q^{\mathbf{Z}}$ and sending t to the point at infinity otherwise.
f) By the q-periodicity of X and Y, ϕ descends to a morphism $\psi : T \to E_q^{\mathrm{an}}$. Show that ψ is an isomorphism of rigid spaces by using that we already know that T is the analytification of a curve of genus one.

The j-invariant of E_q is the classical series $\frac{1}{q} + 744 + 196884q + \cdots$, which has integer coefficients.

Exercise 5.11. Suppose that E is an elliptic curve over K with $|j(E)| > 1$. Show that there is a unique $q \in K^*$ with $|q| < 1$ such that E is isomorphic over an algebraic closure of K to E_q.

One can show using more tools from the arithmetic theory of elliptic curves that the above isomorphism is defined over K precisely when E has split toric reduction. Therefore elliptic curves of split toric reduction over a nonarchimedean field are uniformized by $\mathbf{G}_{m,K}^{\mathrm{an}}$.

Exercise 5.12. Let $q, q' \in K^*$ with $|q| < 1$ and $|q'| < 1$. Prove that E_q and $E_{q'}$ are isogenous if and only if there are positive integers m and n such that $q^m = q'^n$.

There is a higher-dimensional version of the Tate uniformization for abelian varieties with split toric reduction. To prove such a result it no longer suffices to rely on explicit formulas as in Exercise 5.10. One can proceed approximately as follows, at least if R is a dvr (though this restriction turns out to be unnecessary): if A is an abelian variety over K with split toric reduction, let $\mathscr{A}^0$ be the connected component of the identity of the Néron model of A. By assumption the special fiber of $\mathscr{A}^0$ is $\mathbf{G}_{m,k}^n$; rigidity of tori implies that the formal completion of $\mathscr{A}^0$ along the closed fiber is isomorphic to the completion $\hat{G}$ of $\mathbf{G}_{m,R}^n$ along its special fiber. Passing to generic fibers yields an open immersion $i : G \to A^{\mathrm{an}}$, where $G = \mathrm{Sp}(K\langle X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}\rangle)$ is the generic fiber of $\hat{G}$. A Néron-type extension argument in the category of rigid spaces implies that i extends to a morphism $(\mathbf{G}_{m,K}^n)^{\mathrm{an}} \to A^{\mathrm{an}}$, and this turns out to be the quotient morphism uniformizing A^{an}.

Exercise 5.13. Use this sketch to re-prove the Tate uniformization of elliptic curves of split toric reduction (in a less concrete way). See Lemma 1.3 of [3] for the Néron-type extension result.

References

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