Selmer Schemes II, III

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Disclaimer

These lecture slides come with a bibliography at the end. However, there has been no attempt at accurate attribution of mathematical results. Rather, the list mostly contains works the lecturer has consulted during preparation, which he hopes will be helpful for users.
I. De Rham fundamental groups
De Rham fundamental groups

$F$: a finite extension of $\mathbb{Q}_p$.

$X$: a smooth curve over $F$.

$\tilde{X}$: the basechange of $X$ to $\bar{F}$.

$b, x \in X(F)$ viewed sometimes as geometric points:

$$\text{Spec}(\bar{K}) \longrightarrow \tilde{X} \longrightarrow X.$$  

$\mathcal{X}$: a smooth scheme over $\mathcal{O}_F$, the valuation ring of $F$, with good compactification and generic fiber $X$.

$Y$: special fiber of $\mathcal{X}$ over $k = \mathcal{O}_F/m_F$. 
De Rham fundamental groups

The De Rham version is similar to the etale case [Hain, AIK, Kim3]. The relevant category is

$\text{Un}^{DR}(X) \subset \text{Loc}^{DR}(X)$

the category of unipotent vector bundles with (flat) connections, a full subcategory of all bundles with flat connections.

There are fibre functors

$F_b : \text{Un}^{DR}(X) \to \text{Vect}_F,$

$(V, \nabla) \mapsto V_x$

and the objects of interest are

$U^{DR} = U^{DR}(X, b) = \text{Aut}^\otimes(F_b)$

and

$P^{DR}(x) = P^{DR}(X; b, x) = \text{Isom}^\otimes(F_b, F_x)$
De Rham fundamental groups

They can be constructed using universal objects which in turn admit a tautological construction [AIK] using

\[ \text{Ext}_{\text{Loc}^{DR}(X)}^i((V, \nabla), (V', \nabla')) \simeq H_{DR}^i(X, (V, \nabla)^* \otimes (W, \nabla)), \]

where

\[ H_{DR}^i(X, (V, \nabla)) = H^i(X_{\text{Zar}}, V \rightarrow V \otimes_{\mathcal{O}_X} \Omega_X) \]

In particular, it is a projective system

\[ (\mathcal{E}_n^{DR}, \nabla_n), \]

which fit together as

\[ 0 \rightarrow T_n^{DR} \otimes \mathcal{O}_X \rightarrow \mathcal{E}_n^{DR} \rightarrow \mathcal{E}_{n-1}^{DR} \rightarrow 0. \]

Here, \( T_n^{DR} \) is a quotient of \( (H_1^{DR})^{\otimes n} \) as in the étale case.
De Rham fundamental groups

After choosing an element $1 \in \mathcal{E}_{b}^{DR}$ we get the universal property:

Given any object $(V, \nabla_{V})$ in $\text{Un}_{\text{DR}}^{DR}(X)$ together with an element $v \in V_{b}$ (the fiber at $b$), there exists a unique morphism $\phi : (\mathcal{E}_{DR}^{DR}, \nabla) \to (V, \nabla_{V})$ such that $1 \in \mathcal{E}_{b}^{DR} \mapsto v$.

Corollary

$\text{End}(F_{b}) \cong \mathcal{E}_{b}^{DR}$. 
De Rham fundamental groups

Theorem
The pro-algebraic group $U^{DR}(X, b)$ is isomorphic to the group-like elements in $E_b$, while $P^{DR}(X; b, x)$ is isomorphic to the group-like elements in $E_x$.

The universal property gives rise to a map in $\text{Un}(X)$:

$$\Delta : (E^{DR}, \nabla) \longrightarrow (E^{DR}, \nabla) \hat{\otimes} (E^{DR}, \nabla)$$

that takes 1 to $1 \otimes 1$.

Let $A^{DR} = E^{DR}$ be the dual (ind-)bundle. Then $\Delta^*$ gives

$$A_x^{DR} = \text{Hom}(E_x^{DR}, K)$$

the structure of a commutative algebra, and

$$P^{DR}(x) = \text{Spec}(A_x^{DR}).$$
There is a unique decreasing filtration $\mathcal{F}^i$, $i \leq 0$, of $\mathcal{E}^{DR}$ satisfying the following conditions.

1. Griffiths transversality $\nabla (\mathcal{F}^i) \subset \mathcal{F}^{i-1} \otimes \Omega_X$;
2. The induced filtration on $T_n$ coincides with the constant one coming from (co)homology;
3. $1 \in F^0 \mathcal{E}^{DR}$.

This is the Hodge filtration on $\mathcal{E}^{DR}$.

There is an induced Hodge filtration with non-negative degrees on $\mathcal{A}^{DR}$ and $F^1 \mathcal{A}^{DR}$ is an ideal. $F^0 P^{DR}(x)$ is the defined to be the zero set of $F^1 \mathcal{A}^{DR}_x$. It is a torsor for $F^0 U^{DR}$, which is a subgroup of $U^{DR}$. 
De Rham fundamental groups: Hodge filtration

This is an aspect of the fact that the action of $U^{DR}$ on $P^{DR}(x)$ is compatible with the Hodge filtration. The action map

$$P^{DR}(x) \times U^{DR} \rightarrow P^{DR}(x)$$

corresponds to a co-action map

$$\mathcal{A}_x^{DR} \rightarrow \mathcal{A}_x^{DR} \otimes \mathcal{A}_b^{DR}$$

This is compatible with the Hodge filtration.

The choice of a point $p \in F^0P^{DR}(x)$ gives an algebra homomorphism $\mathcal{A}_x^{DR} \rightarrow F$ which kills $F^1\mathcal{A}_x^{DR}$, which is hence a map of Hodge structures.
Thus, we get an isomorphism

\[ \mathcal{A}_x^{DR} \cong \mathcal{A}_b^{DR} \]

that is compatible with the Hodge filtration. A dimension count then shows that

\[ F^1 \mathcal{A}_x^{DR} \cong F^1 \mathcal{A}_b^{DR}, \]

and hence,

\[ \mathcal{A}_x^{DR} / F^1 \mathcal{A}_x^{DR} \cong \mathcal{A}_b^{DR} / F^1 \mathcal{A}_b^{DR}, \]

giving us

\[ F^0 U^{DR} \cong F^0 P^{DR}(x). \]
In the local case, the \((k-\text{linear})\) Frobenius \(\phi\) of the special fibre \(Y\) acts on the category \(\text{Un}^{\text{DR}}(X)\) [Deligne, Besser].

Write \(\mathcal{X} = \bigcup_i U_i\) so that \(U_i\) is a smooth lift of \(U_i \otimes k\). Choose local lifts \(\phi_i\) on \(U_i\) of the Frobenius on \(U_i \otimes k\).

Then given a bundle with connection \((V, \nabla)\), we consider the local pull-backs \((\phi_i^*(V|_{U_i}), \phi_i^*(\nabla))\). The connection allows us to patch these together canonically to give us \(\phi^*(V, \nabla)\).

In particular,

\[
(\mathcal{E}^{\text{DR}}, \nabla, 1) \longrightarrow (\phi^*\mathcal{E}^{\text{DR}}, \phi^*\nabla, \phi^*1),
\]

Get compatible actions on \(U^{\text{DR}}(V, b)\) and \(P^{\text{DR}}(X; b, x)\).
On $T_n$, agrees with the action induced by the isomorphism

$$H^1_{DR}(X) \cong H^1_{crys}(Y).$$

Hence, the eigenvalues are the same as the ones coming from étale cohomology.

**Theorem**

*There is a unique Frobenius invariant element $p^{cr}_{b,x}$ in $P^{DR}(X, b, x)$.***
Lemma
The Lang map $L(\phi) : U^{DR} \to U^{DR}$ that sends $u$ to $u\phi^{-1}(u)$ is a bijection.

In particular, the identity is the only element fixed by $\phi$.

Proof.
The eigenvalues of $\phi$ on $T^{DR}_n = U^{DR,n}/U^{DR.n+1}$ are all different from 1.

Proof of theorem.
Choose $p \in P^{DR}$. Then there is a unique $u \in U^{DR}$ such that $\phi(p) = pu$. Write $u = v\phi(v^{-1})$. Then

$$\phi(pv) = pv.$$ 

Uniqueness comes from the fact that if $p$ is fixed, no $pu$ will be fixed for $u \neq e$. 

\[\square\]
Better to think in terms of *crystalline fundamental groups*: Given a point \( y \in Y(k) \), define on \( \text{Un}(X)^{DR} \) the fibre functor

\[
(V, \nabla) \mapsto V([y])^{\nabla=0},
\]

the flat sections of \( V \) over the tube \( [y] \) of \( y \), the analytic space of points that reduce to \( y \).

Then for \( x, x' \in [y] \), \( p_{x,x'}^{cr} \) is given by the diagram

\[
\begin{array}{ccc}
V([y])^{\nabla=0} & \xrightarrow{\sim} & V_x \\
\Uparrow & & \downarrow \\
V([y])^{\nabla=0} & \xrightarrow{\sim} & V_{x'}
\end{array}
\]
De Rham fundamental groups: crystalline structures

This is supplemented by an isomorphism

\[ p^\text{cr}_{yy'} : V(\lfloor y \rfloor)^{\nabla=0} \cong V(\lfloor y' \rfloor)^{\nabla=0} \]

for \( y, y' \in Y(k) \) called Coleman integration [Besser]. The computation of this is Kedlaya’s theory.
De Rham moduli spaces

The space of torsors for $U^{DR}$ that have compatible Frobenius and Hodge filtration are classified by

$$U^{DR}/F^0.$$  

Given a torsor $T$, choose elements $t^{cr} \in T$ and $t^H \in F^0 T$. Then

$$t^H = t^{cr} u_T^{cr}.$$  

The element $u_T^{cr}$ is independent of the choice of $t^H$ up to multiplication by $F^0 U^{DR}$ on the right, giving us a well-defined element

$$[u_T^{cr}] \in U^{DR}/F^0.$$
We will give an explicit description for $X$ affine [Kim3].

We first choose

$$\alpha_1, \alpha_2, \cdots, \alpha_m,$$

global algebraic differential forms representing a basis of $H^1_{DR}(X)$. Thus, $m = 2g + s - 1$, where $s$ is the number of missing points.
De Rham fundamental groups

Consider the algebra
\[ F \langle A_1, \ldots, A_m \rangle \]
generated by the symbols \( A_1, A_2, \ldots, A_m \). Thus, it is the tensor algebra of the \( F \)-vector space generated by the \( A_i \). Let \( I \) be the augmentation ideal.

The algebra \( F \langle A_1, \ldots, A_m \rangle \) has a natural comultiplication map \( \Delta \) with values \( \Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i \).

Now let
\[ E_n = F \langle A_1, \ldots, A_m \rangle / I^{n+1} \]
and take the completion
\[ E := \varprojlim F \langle A_1, \ldots, A_m \rangle / I^n \]
\( \Delta \) extends naturally to a comultiplication \( E \to E \hat{\otimes} E \).
De Rham fundamental groups

\( \mathcal{E} \): pro-unipotent pro-vector bundle \( E \otimes \mathcal{O}_X \) with the connection \( \nabla \) determined by

\[
\nabla \mathcal{E} f = df - \Sigma_i A_i f \alpha_i
\]

for sections \( f : X \to E \).

There is an element \( 1 \in \mathcal{E}_b = E \).

**Theorem**

*There is a unique isomorphism*

\[
(\mathcal{E}, \nabla_{\mathcal{E}}, 1) \cong (\mathcal{E}^{DR}, \nabla, 1)
\]

*It is compatible with the comultiplication on either side.*
De Rham fundamental groups

The theorem is an easy consequence of

Lemma
Let \((V, \nabla)\) be a unipotent bundle with flat connection on \(X\) of rank \(r\). Then there exist strictly upper-triangular matrices \(N_i\) such that

\[
(V, \nabla) \simeq (\mathcal{O}_X^r, d + \sum \alpha_i N_i)
\]
De Rham fundamental groups

The isomorphism

\[ \mathcal{E}^{DR}_{\partial}(\mathcal{V}) \triangleright \mathcal{E}^{DR}_{\partial} \sim \mathcal{E}^{DR}_{\partial}, \]

\[ \mathcal{E}^{DR}_{\partial} \sim \mathcal{E}^{DR}_{\partial} \]

can be constructed locally by solving differential equations.

Let

\[ f = \sum_{w} f_{w}[w] \]

be a section of \( \mathcal{E} \), where the \([w]\) are words in the \( A_i \), and \( f(b) = 1 \).

Then the flatness condition is

\[ df = \sum_{w} \sum_{i} f_{w}\alpha_{i}[A_{i}w], \]
De Rham fundamental groups

This is

\[ df_{A_i \omega} = f_\omega \alpha_i \]

for all \( \omega \) and \( i \).

We solve this iteratively:

\[ f_{A_i}(z) = \int_{b}^{z} \alpha_i. \]

This can be constructed as a power series with initial condition \( f_{A_i}(x) = 0 \).

We continue

\[ f_{A_j A_i}(z) = \int_{b}^{z} f_{A_i} \alpha_j, \]

and so on. Thus, the components of \( f \) become iterated integrals.
De Rham fundamental groups

Having solved the equation with initial condition 1, get \( p_{bx}^{cr} \) for \( v \in \mathcal{E}_b^{DR} \) by

\[
p_{bx}^{cr}(v) = f(x)v.
\]

For general \( x \), the components of \( p_{bx}^{cr} \) give the definition of iterated integrals.

The shuffle identities for iterated integrals

\[
\int_b^z \alpha_1 \alpha_2 \cdots \alpha_k \int_b^z \alpha_{k+1} \alpha_{k+2} \cdots \alpha_n = \sum_{\sigma} \int_b^z \alpha_{\sigma(1)} \alpha_{\sigma(2)} \cdots \alpha_{\sigma(n)}
\]

with the sum running over \((k, n - k)\) shuffles of \( \{1, 2, \ldots, n\} \) follow from the group-like nature of \( p_{b,z}^{cr} \).
De Rham fundamental groups

Another way to say this is that

\[ A_{z}^{DR} = F[\phi_{w}] \]

the vector space generated by \( \phi_{w} \) such that \( \phi_{w}[w'] = \delta_{ww'} \). The algebra structure is given by

\[ \phi_{w}\phi_{w'} = \sum_{\sigma} \phi_{\sigma(ww')} , \]

where again the \( \sigma \) run over shuffles. The iterated integral identity is the fact that

\[ p_{b,z}^{cr} : A_{z}^{DR} \rightarrow F \]

is an algebra homomorphism.
De Rham fundamental groups

**Theorem**

The map

$$j^{DR} : X(F) \longrightarrow U^{DR}/F^0$$

has the property that $j^{DR}(\{y\})$ is Zariski dense for each $y \in Y$.

The idea is to show that all iterated integrals are algebraically independent using transcendental methods.

Hence, as we increase $n$, the coordinates of the map

$$j^{DR} : X(F) \longrightarrow U^{DR}_n/F^0$$

keep giving genuinely new analytic functions.
Selmer schemes III
I. Preliminary remarks on gauge theory
Gauge theory

\( M \): connected orientable manifold.

\( G \): Lie group.

\( \mathcal{A}(M, G) \): the space of principal \( G \)-bundles with connections.

In gauge theory, study various functions on \( \mathcal{A}(M, G) \), for example,

\[
YM(P, \nabla) = \int_M \| F_\nabla \wedge *F_\nabla \|^2
\]
on a four-manifold, or

\[
CS(P, \nabla) = \int_M \text{Tr}(A \wedge dA + (2/3)A \wedge A \wedge A).
\]
on a three-manifold.
Gauge theory

The critical points of such functions will form moduli spaces

$\mathcal{S}(M, G)$

of classical solutions.

These can be studied for their own sake, or via the analogy

$\mathcal{S}(M, G) \sim H^1(M, G)$

as non-abelian cohomological invariants of $M$.

These have seen spectacular applications in geometry, Donaldson theory, Simpson theory, Seiberg-Witten theory, TQFT, ... (e.g. [Donaldson, Witten])
In recent decades, geometers have become interested in formal integrals like
\[ \int_{A/G} e^{-kCS(A)} dA \]
or
\[ \int_{A/G} Hol_{K_1,R_1}(A) Hol_{K_2,R_2}(A) \cdots Hol_{K_m,R_m}(A) e^{-kCS(A)} dA, \]
where the $K_i$ are closed curves in $M$ and $Hol_{K_i,R_i}(A)$ denotes the holonomy in some representation $R_i$ of $G$.

Would like to develop such tools for the geometric study of arithmetic schemes.
II. Geometry of non-abelian cohomology
Non-abelian cohomology functors

$X/\mathbb{Q}$: a smooth curve and $p > 2$ a place of good reduction.

$U = U(\bar{X}, b)$, the $\mathbb{Q}_p$-prounipotent étale fundamental group.

$U_n = U/U^{n+1}$.

$G$: either the group $\text{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ or $G_T = \text{Gal}(\mathbb{Q}_T/\mathbb{Q})$, where $\mathbb{Q}_T$ is the maximal extension of $\mathbb{Q}$ unramified outside a finite set $T$ of primes. We assume that $T$ contains $\infty, 2, p$ and all primes of bad reduction.
Non-abelian cohomology functors

[Kim1]
We define a functor of $\mathbb{Q}_p$-algebras

$$R \mapsto H^1(G, U_n(R)) := U_n(R) \backslash Z^1(G, U_n(R)).$$

The $H^1$ refers to continuous cohomology: $Z^1$ denotes the continuous functions

$$f : G \longrightarrow U(R)$$

such that $f(g_1 g_2) = f(g_1) g_1(f(g_2))$ on which $U_n(R)$ acts via

$$f^u(g) = uf(g)g(u^{-1}).$$
Non-abelian cohomology functors

The $G$-action on $U_n(R)$ is defined by identifying

$$U_n \cong^{\log} L_n := \text{Lie}(U_n).$$

In fact, it is often good to think of $U_n$ as being $L_n$ with group law defined by the BCH formula:

$$X \cdot Y = X + Y + (1/2)[X, Y] + (1/12)[X, [X, Y]] - (1/12)[Y, [Y, X]] + \cdots.$$  

(Formula for $\log(\exp(X) \exp(Y))$.)

Then $U_n(R) = L_n \otimes R.$
Non-abelian cohomology functors

The topology in $U_n(R)$ is defined by using

$$U_n \cong \mathbb{A}^N,$$

which gives

$$U_n(R) \cong R^N.$$  

We give $R^N$ the inductive limit topology of finite-dimensional $\mathbb{Q}_p$-subspaces. (This definition works also for all affine schemes.)

On the abelian pieces $U^n/U^{n+1}$, the same definition of $H^1$ applies, but we can also define $H^2$.

Proposition

$$H^i(G, U^n/U^{n+1}(R)) \cong H^i(G, U^n(\mathbb{Q}_p)/U^{n+1}(\mathbb{Q}_p)) \otimes R.$$  

That is, the functor of $R$ can be represented by the finite-dimensional $\mathbb{Q}_p$-vector space $H^i(G, U^n(\mathbb{Q}_p)/U^{n+1}(\mathbb{Q}_p))$. 
Theorem

The functor

\[ R \mapsto H^1(G, U_n(R)) \]

is represented by an affine \( \mathbb{Q}_p \)-scheme of finite type.

The scheme represents principal \( U_n \)-bundles with continuous \( G \) action:

The \( R \)-points are principal \( (U_n)_R \) bundles

\[ P \longrightarrow \text{Spec}(R), \]

with functorial continuous action of \( G \) on \( P(S) \) for any \( R \)-algebra \( S \).
The proof is by induction on $n$ using the exact sequence

$$
0 \to H^1(G, U^n/U^{n+1}(R)) \to H^1(G, U_n(R)) \to H^1(G, U_{n-1}(R)) \to H^2(G, U^n/U^{n+1}(R)).
$$

That is, once $H^1(G, U_{n-1})$ is representable, $\delta$ is a map of schemes. The exact sequence means that $H^1(G, U_n)$ defines a $H^1(G, U^n/U^{n+1})$-torsor over Ker($\delta$), which then must be represented by

$$
\text{Ker}(\delta) \times H^1(G, U^n/U^{n+1}).
$$
Non-abelian cohomology functors

In the local case, define also

\[ R \mapsto H^1(G, U_n(B_{\text{cris}} \otimes R)), \]

and

\[ H^1_f(G, U_n) = \text{Ker}(H^1(G, U_n) \longrightarrow H^1(G, U_n(B_{\text{cris}}))), \]

which is a subscheme by induction on \( n \):

\[
\begin{array}{c}
0 \longrightarrow H^1(G, U^n/U^{n+1}) \longrightarrow H^1(G, U_n) \longrightarrow H^1(G, U_{n-1}) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow H^1(G, U^n/U^{n+1}(B_{\text{cris}})) \longrightarrow H^1(G, U_n(B_{\text{cris}})) \longrightarrow H^1(G, U_{n-1}(B_{\text{cris}}))
\end{array}
\]
Non-abelian cohomology functors

\( H^1_f(G_p, U_n) \) represents torsors that have a \( G_p \)-invariant point in \( U_n(B_{cris}) \).

We have the localisation

\[
H^1(G_T, U_n) \longrightarrow H^1(G_p, U_n)
\]

using which we define \( H^1_f(G_T, U_n) = \text{loc}_p^{-1}(H^1_f(G_p, U_n)) \).

Thus, we get a diagram

\[
\begin{array}{ccc}
X(\mathbb{Z}) & \longrightarrow & X(\mathbb{Z}_p) \\
\downarrow & & \downarrow \\
H^1_f(G_T, U_n) & \longrightarrow & H^1_f(G_p, U_n)
\end{array}
\]

The bottom arrow is a map of schemes since it represents a map of functors. It is a *computable replacement* for \( X(\mathbb{Z}) \subset X(\mathbb{Z}_p) \),
De Rham moduli spaces

The reason $X(\mathbb{Z}_p)$ maps to $H^1_f$ is because of the non-abelian $p$-adic Hodge theory isomorphism:

$$P^\text{et}_n(x)(B_{cr}) \cong P^\text{DR}_n(x)(B_{cr}) \cong B^N_{cr}.$$  

The first isomorphism respects all structures, while the second is Galois equivariant, showing the existence of an invariant point.
II. The fundamental diagram
The fundamental diagram

[Kim1, Kim2, Kim3]

Given $T$ a crystalline torsor for $U$, the trivialisations of $T(B_{cr})$ form a torsor $D(T)$ for $U^{DR}$.

Lemma

$$T \mapsto D(T)$$

defines an isomorphism

$$H^1_f(G_p, U_n) \cong U_n^{DR}/F^0.$$
The fundamental diagram

\[ X(\mathbb{Z}) \rightarrow X(\mathbb{Z}_p) \]

\[ H^1_f(G_T, U_n) \rightarrow H^1_f(G_p, U_n) \cong U^{DR}_n / F^0 \]

The isomorphism on the right comes from the construction of an inverse using the fundamental exact sequence of \( p \)-adic Hodge theory:

\[ 0 \rightarrow \mathbb{Q}_p \rightarrow B^{\phi=1}_{cr} \oplus B^+_\text{DR} \rightarrow B_{\text{DR}} \rightarrow 0. \]
The fundamental diagram

From this, we get

\[\frac{U(B_{DR})}{U(B_{DR}^+)} \rightarrow H^1(G, U) \rightarrow H^1(G, U(B_{cr}^\phi))].\]

For \(U\), we get an equality between

\[H^1_e(G, U) = \text{Ker}[H^1(G, U) \rightarrow H^1(G, U(B_{cr}^\phi))].\]

and

\[H^1_f(G, U) = \text{Ker}[H^1(G, U) \rightarrow H^1(G, U(B_{cr}))].\]


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