AWS 2020: Geometric quadratic Chabauty

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1 Course description

The quadratic Chabauty method was developed by Kim, Balakrishnan, Besser, Dogra and Müller, and extended in [BDMTV], for finding all rational points on a curve C of genus at least two, provided that $r < g + \rho - 1$. Here, r is the rank of $J(\mathbb{Q})$, with J the jacobian of C, g is the genus of C, and ρ is the Picard number (over $\mathbb{Q}$) of J.

The course has two aims. To describe the quadratic Chabauty method in terms of algebraic geometry only: models over the integers of line bundles on J. And to give an algorithm that can verify, in each given instance where $r < g + \rho - 1$, that the list of known rational points is complete. The course does not aim at effective or uniform finiteness results for *classes of curves.*

The course will follow the preprint [E-L], providing more background where or when needed. The number $g + \rho - 1$ is the dimension of a product T of $\rho - 1$ principal $\mathbb{G}_{\mathrm{m}}$-bundles on J. As in the classical (linear) Chabauty method, we are intersecting, for p a prime number, but now in $T(\mathbb{Q}_p)$ in stead of in $J(\mathbb{Q}_p)$, the closure of $T(\mathbb{Z})$, which has dimension $\leq r$, with $C(\mathbb{Q}_p)$.

Planning of the lectures

The following planning is preliminary, and will be adapted as the course goes on. One idea is to carry the example in Section 8 along through all the lectures.

1. Section 2.
2. Sections 3 and 4.
3. Section 4.
4. Sections 6 and 7 (not in detail).

2 Projects and required background

2.1 Translation from the geometric approach to the fundamental group theoretical approach

The aim of this project[1] is to relate the fundamental group approach to quadratic Chabauty as in [BDMTV] to the geometric method in [E-L]. It is claimed that experts more or less know how to do this. It is also true that the authors of the articles just mentioned do *not* know the details, but are really interested in them.

[1]In this version (2020, March 9) some misconceptions of me (Bas) about the real Arakelov picture are corrected, following remarks by Netan Dogra.

Our starting point is the geometric approach as in [E-L], and, more precisely, the diagram (2.12) on page 5 (see there for details):

$$\begin{tikzcd} & T \arrow[r] \arrow[d] & P^{\times,\rho-1} \arrow[d] \\ U \arrow[ur, "\widetilde{j_b}"] \arrow[r, "j_b"] & J \arrow[r, "(\mathrm{id}, m\cdot\circ \mathrm{tr}_{c_i}\circ f_i)_i"] & J \times (J^{\vee 0})^{\rho-1}\,. \end{tikzcd}$$

In [M-B], Theorem 5.4, Moret-Bailly shows how P is naturally equipped with metrics on its fibres over $J(\mathbb{C}) \times J^\vee(\mathbb{C})$, and that for (x, y) in $(J \times J^{\vee 0})(\mathbb{Z})$, the Arakelov degree of $(x, y)^*P$ is the Néron-Tate height of (x, y).

Now T is the product of $\rho-1$ principal $\mathbb{G}_\mathrm{m}$-bundles T_i on J. Their associated line bundles $\mathcal{L}_i$ are pullbacks of P and so have natural metrics over $J(\mathbb{C})$. The pullbacks of the $\mathcal{L}_i$ to U are trivial (uniquely up to signs). The squares of the norms of the trivialising sections are functions $g_i\colon U(\mathbb{C}) \to \mathbb{R}_{>0}$ with $(2\pi i)^{-1}\partial\overline{\partial}\log g_i$ equal to the pullback by $(\mathrm{id}, m\cdot \circ \mathrm{tr}_{c_i} \circ f_i)$ of the curvature form of the metric of P on $J(\mathbb{C}) \times J^\vee(\mathbb{C})$. The knowledge of $(2\pi i)^{-1}\partial\overline{\partial}\log g_i$ together with the value of g_i at b (see (7.8) and (7.2) in [E-L]) determine the g_i. For any u in $U(\mathbb{Z})$ the Néron-Tate height of $(j_b(u), m(c_i + f_i(j_b(u))))$ is equal to the Arakelov height of $j_b(u)$ with respect to $\mathcal{L}_i$ with its metric, and to the Arakelov degree of $(j_b(u), m(c_i + f_i(j_b(u))))^*P$, and the trivialising section makes this Arakelov degree equal to $(-1/2)\log g_i(u)$. This gives a bound on the absolute value of the Néron-Tate height of $(j_b(u), m(c_i + f_i(j_b(u))))$. This might help in determining the x in $J(\mathbb{Q})$ that are in $U(\mathbb{Z})$ but note that the functions $J(\mathbb{Q}) \to \mathbb{R}$ that send x to the Néron-Tate height of $(x, m(c_i + f_i(x)))$ are not definite.

So, following Chabauty, one tries a p-adic approach. Here this means considering, for some chosen prime p, p-adic valued heights and Arakelov theory. This is done in [M-T] for abelian varieties, using biextensions, and in [C-G] for jacobians. The idea is that the product formula must be preserved and that the analytic functions at the archimedean places are replaced by p-adic analytic functions at the p-adic places. For example, the $\mathbb{R}$-valued adele norm on the ideles of $\mathbb{Q}$, $x \mapsto \|x\| = \prod_v |x_v|_v$, is trivial on $\mathbb{Q}^\times$. For $x \in \mathbb{R}^\times$ we have $|x|_\infty = x\cdot\mathrm{sign}(x)$. The factor x can be moved, for $x \in \mathbb{Q}^\times$, to any p-adic place of our choice. So, for a prime p, we obtain the $\mathbb{Q}_p^\times$-valued adele norm $x \mapsto (x_p\cdot|x_p|_p)\cdot\mathrm{sign}(x_\infty)\cdot \prod_{v\notin\{p,\infty\}} |x_v|_v$, also trivial on $\mathbb{Q}^\times$. Up to a sign, it corresponds via class field theory for $\mathbb{Q}$ to the p-adic cyclotomic character. The main point is that at all places other than p and ∞, nothing has changed.

We are now already very close to § 1.4 of [BDMTV], and it should not be very hard to get a precise translation.

For the subsequent interpretation in [BDMTV] of everything in terms of fundamental groups, we note that the embedding $\widetilde{j_b}\colon C_\mathbb{Q} \to T_\mathbb{Q}$ induces a morphism of fundamental groups. The complex uniformisation of $P^\times(\mathbb{C})$ (see [B-E, §4]) gives the structure of $\pi_1(P^\times(\mathbb{C}))$; it is a non-abelian extension of $\pi_1(J(\mathbb{C}) \times J(\mathbb{C}))$ by $\mathbb{Z}$. So, apparently, one has to study p-adic local systems on $T_\mathbb{Q}$.

Required background.

Basic knowledge of the algebraic geometry in [E-L], mainly over $\mathbb{C}$ and over $\mathbb{Q}$. Some Arakelov height theory (see [M-B], [H-S], and [H]) and p-adic height theory ([M-T] and [C-G])).

For the passage from p-adic heights to fundamental groups, some working knowledge of Galois cohomology and etale cohomology (see [Po]), algebraic de Rham cohomology (see `https://en.wikipedia.org/wiki/Khler_differential`), knowledge in abelian and non-abelian p-adic Hodge theory (see the references in [BDMTV]).

2.2 Comparing computations with participants to Jennifer Balakrishnan's project 2: modular curves $X_0(n)^+$.

The aim here is to apply the geometric quadratic Chabauty method to the curves $X_0(n)^+$ mentioned in Jennifer Balakrishnan's project, and then to compare the whole process with the participants of that project.

We hope that this comparison gives some insight in running times on both sides, actually even for linear Chabauty (as treated in David Zureick-Brown's lectures): Coleman integrals on $C(\mathbb{Q}_p)$ versus computations in $J(\mathbb{Z}/p^2\mathbb{Z})$.

Here one can build on Guido Lido's example (Section 8 in [E-L]) and his code in cocalc, to be found with [E-L]. It may be that at the time of the School the example $X_0(73)^+$ will be available.

Required background

Section 8 (and therefore most of the other sections as well) of [E-L]. Some knowledge of modular curves, see [D-S].

2.3 Generalisation of the geometric quadratic Chabauty method to number fields

This generalisation has already been carried out for bielliptic curves of genus 2 in [BBBM]. It is interesting to see, at first theoretically, how the methods of [E-L] can be generalised to number fields. The first idea is to use Weil restriction, to reduce to geometry over $\mathbb{Q}$.

Required background

Section 2 of [E-L].

References

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