1. Course Outline

Given a smooth projective curve $X/\mathbb{Q}$, one aim of Kim’s nonabelian Chabauty program [Kim09, Kim10a, Kim10b] is to determine $X(\mathbb{Q})$ algorithmically. This course will highlight the computational aspects of the quadratic Chabauty method [BD18, BD17, BDM+19], and in particular, describe algorithms used to compute the finite set of $p$-adic points $X(\mathbb{Q}_p)_{\geq 2}$ in certain cases where

$$r < g + \rho - 1,$$

where $g$ is the genus of $X$, $\rho$ is the Picard number of the Jacobian $J$, and $r = \text{rk} J(\mathbb{Q})$. Here is a provisional outline of the lectures in this course.

Lecture I: The basic tools. Start by carrying out the linear algebra of explicit Chabauty–Coleman for curves over $\mathbb{Q}$, with Coleman integration as a black box. Describe how the Chabauty–Coleman diagram generalizes and motivate the presence of iterated Coleman integrals. Discuss Coleman integration [Col85, Bes12] and give preliminaries for explicit Coleman integration, starting with the $p$-adic point-counting algorithm of Kedlaya and Tuitman [Ked01, Ked07, Tui16, Tui17].

Lecture II: The basic tools, continued. Give algorithms for computing Coleman integrals of differentials of the first and second kind on curves [BBK10, BT17]. Describe algorithms to compute Coleman–Gross local $p$-adic heights [CG89, BB12], and in particular, Coleman integrals of 1-forms of the third kind on curves.


Lecture IV: Examples. Apply the computation of the Nekovář height in a collection of examples to determine $X(\mathbb{Q})$. This could include bielliptic genus 2 curves, modular curves of genus 3 with real multiplication, and curves with few rational points. Discuss where the current frontier is and what remains to be done.

2. Projects

Below are some ideas for possible projects:

(1) **Coleman integration for curves over number fields and a Chabauty–Coleman solver.** The goals of this project would be to give an algorithm to compute Coleman integrals on curves over number fields, implement the algorithm, and use this to give a Chabauty-Coleman solver for curves over number fields that would take as input a genus $g$ curve $X$ defined over a number field $K$ with $r = \text{rk} J(K) < g$, a prime $p$ of good reduction, and $r$ generators of the Mordell–Weil group modulo torsion and output the set $X(K_p)$.  

Suggested reading: Coleman integration.
(2) **Quadratic Chabauty on modular curves** $X_0(N)^+$. Galbraith [Gal96, Gal99, Gal02] has constructed models for all modular curves $X_0(N)^+ = X_0(N)/w_N$ of genus $\leq 5$ (with the exception of $N = 263$) and has conjectured that he has found all exceptional points on these curves. This project will use quadratic Chabauty to prove as much as possible about Galbraith’s conjecture. Another goal is to investigate whether we can use $p$-adic Gross-Zagier to carry out quadratic Chabauty for $X_0(N)^+$, starting with the case of such curves of genus 2.

Suggested reading: Modular curves, $p$-adic heights, $p$-adic $L$-functions.

(3) **Quadratic Chabauty and Kim’s conjecture.** When $X/\mathbb{Q}$ is a genus $g$ curve with $r = \text{rk} J(\mathbb{Q}) = g - 1$, then typically the set of $p$-adic points $X(\mathbb{Q}_p)_1$ cut out by the Chabauty-Coleman method strictly contains $X(\mathbb{Q})$. In this project, we will first give an algorithm to compute the quadratic Chabauty set $X(\mathbb{Q}_p)^2$ under these hypotheses. Then we will investigate whether the quadratic Chabauty set, which satisfies

$$X(\mathbb{Q}) \subset X(\mathbb{Q}_p)^2 \subset X(\mathbb{Q}_p)_1 \subset X(\mathbb{Q}_p),$$

is equal to $X(\mathbb{Q})$. (See [Bia19] for the case of integral points on punctured elliptic curves.) If $X(\mathbb{Q}) \neq X(\mathbb{Q}_p)^2$, we would like to characterize the points in $X(\mathbb{Q}_p)^2 \setminus X(\mathbb{Q})$. This project could be carried out on a database of genus 2 and 3 curves [The19].

Suggested reading: Chabauty-Coleman method, $p$-adic heights.

For the computational part of Projects 1 and 3, we will use the computer algebra system **Magma**. For Project 2, **Magma** would be useful, but restricting to the case of hyperelliptic curves would also be very interesting (and likely more tractable, from the point of view of determining Mordell–Weil ranks unconditionally), and in this case, we could use **SageMath**.


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