1. Projects: Propagating the Iwasawa main conjecture via congruences

1.1. Goal of these projects. Let \( f, g \in S_k(\Gamma_0(N)) \) be normalized eigenforms (not necessarily newforms) of weight \( k \geq 2 \), say with rational Fourier coefficients \( a_n, b_n \in \mathbb{Q} \) for simplicity, and assume that

\[
f \equiv g \pmod{p}
\]

in the sense that \( a_n \equiv b_n \pmod{p} \) for all \( n > 0 \). Roughly speaking, the goal of these projects is to study how knowledge of the Iwasawa main conjecture for \( f \) can be “transferred” to \( g \).

For \( k = 2 \) and primes \( p \mid N \) of ordinary reduction, such study was pioneered by Greenberg–Vatsal [GV00], and in these projects we will aim to extend some of their results to:

- non-ordinary primes;
- certain anticyclotomic settings;
- (more ambitiously) some of the “residually reducible” cases which eluded the methods of [GV00], with applications to the \( p \)-part of the BSD formula in ranks 0 and 1.

1.2. The method of Greenberg–Vatsal. Before jumping into the specifics of each of those settings, let us begin with a brief outline of the method of Greenberg–Vatsal (which is beautifully explained in [GV00, §1]). Let \( F_{\infty}/F \) be a \( \mathbb{Z}_p \)-extension of a number field \( F \), and identify the Iwasawa algebra \( \mathbb{Z}_p[[\text{Gal}(F_{\infty}/F)]] \) with the one-variable power series ring \( \Lambda = \mathbb{Z}_p[[T]] \) in the usual fashion.

Recall that Iwasawa’s main conjecture for \( f \) over \( F_{\infty}/F \) posits the following equality between principal ideals of \( \Lambda \):

\[
(L_p^{\text{alg}}(f))^g \equiv (L_p^{\text{an}}(f)),
\]

where

- \( L_p^{\text{alg}}(f) \in \Lambda \) is a characteristic power series of a Selmer group for \( f \) over \( F_{\infty}/F \);
- \( L_p^{\text{an}}(f) \in \Lambda \) is a \( p \)-adic \( L \)-function interpolating critical values for \( L(f/F,s) \) twisted by certain characters of \( \text{Gal}(F_{\infty}/F) \).

By the Weierstrass preparation theorem, we may uniquely write

\[
I_p^{\text{alg}}(f) = \mu^{\text{alg}}(f) \cdot Q^{\text{alg}}(f) \cdot U,
\]

with \( \mu^{\text{alg}}(f) \in \mathbb{Z}_{\geq 0} \), \( Q^{\text{alg}}(f) \in \mathbb{Z}_p[T] \) a distinguished polynomial, and \( U \in \Lambda^\times \) an invertible power series. Letting

\[
\lambda^{\text{alg}}(f) := \deg Q^{\text{alg}}(f),
\]

and similarly defining \( \mu^{\text{an}}(f) \) and \( \lambda^{\text{an}}(f) \) in terms \( L_p^{\text{an}}(f) \), the strategy of [GV00] is based on the following three observations:

**O1.** The equality (1.1) amounts to having:

(1) \( (L_p^{\text{alg}}(f)) \supseteq (L_p^{\text{an}}(f)) \),

(2) \( \mu^{\text{alg}}(f) = \mu^{\text{an}}(f) \),

(3) \( \lambda^{\text{alg}}(f) = \lambda^{\text{an}}(f) \).

We shall place ourselves in a situation where one expects that \( \mu^{\text{alg}}(f) = \mu^{\text{an}}(f) = 0 \).

**O2.** For \( \Sigma \) any finite set of primes \( \ell \neq p, \infty \), the equality (1.1) is equivalent to the equality

\[
(L_p^{\Sigma, \text{alg}}(f))^g \equiv (L_p^{\Sigma, \text{an}}(f)),
\]

where \( L_p^{\Sigma, \text{alg}}(f) \) and \( L_p^{\Sigma, \text{an}}(f) \) are the “imprimitive” counterparts of \( L_p^{\text{alg}}(f) \) and \( L_p^{\text{an}}(f) \) obtained (roughly speaking) by relaxing the local conditions/removing the Euler factors at the primes \( \ell \in \Sigma \).

**O3.** For appropriate \( \Sigma \), the objects involved in (1.2) are well-behaved under congruences. Letting \( \mu_{p,\Sigma}^{\text{alg}}(f), \lambda_{p,\Sigma}^{\text{alg}}(f) \), etc. be the obvious invariants from the above discussion, this translates into:

\[
(L_p^{\Sigma, \text{alg}}(f))^g \equiv (L_p^{\Sigma, \text{an}}(f)),
\]
Expectation 1. Assume that $f \equiv g \pmod{p}$, and let $\star \in \{\text{alg, an}\}$. If $\mu_\Sigma^\star(f) = 0$, then $\mu_\Sigma^\star(g) = 0$ and $\lambda_\Sigma^\star(f) = \lambda_\Sigma^\star(g)$.

Now, if we are given $f \equiv g \pmod{p}$ and the divisibilities
\[
(1.3) \quad (L_p^\text{alg}(f)) \supseteq (L_p^\text{an}(f)) \quad \text{and} \quad (L_p^\text{alg}(g)) \supseteq (L_p^\text{an}(g)),
\]
we see that the equivalence of O2 combined with Expectation 1 yields the implication
\[
(1.4) \quad (L_p^\text{alg}(f)) = (L_p^\text{an}(f)) \implies (L_p^\text{alg}(g)) = (L_p^\text{an}(g)).
\]
Note that this has interesting applications. Indeed, if for example the residual representation $\bar{\rho}_f$ is absolutely irreducible, then one can hope to establish (1.3) by an Euler/Kolyvagin system argument. Proving the opposite divisibility (either via Eisenstein congruences, or via a refined Euler/Kolyvagin system argument) often requires additional ramification hypotheses on $\bar{\rho}_f$ relative to the level of $f$ (see below for specific examples), a restriction that could be ultimately removed thanks to (1.4).

1.3. On the cyclotomic main conjectures for non-ordinary primes. Here we let $F_\infty/F$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, let $p \nmid N$ be a non-ordinary prime for $f \in S_k(\Gamma_0(N))$, and let $\alpha, \beta$ be the roots of the $p$-th Hecke polynomial of $f$. In this setting, Lei–Loeffler–Zerbes [LLZ10], [LLZ11], formulated1 “signed” main conjectures:
\[
(1.5) \quad (L_p^\alpha(f)) \equiv \text{Char}_\Lambda(\text{Sel}_f(f)^\vee), \quad (L_p^\beta(f)) \equiv \text{Char}_\Lambda(\text{Sel}_f(f)^\vee),
\]
where $\text{Sel}_f(f)$ and $\text{Sel}_f(f)$ are Selmer groups cut out by local condition at $p$ more stringent than the usual ones, and $L_p^\alpha(f), L_p^\beta(f) \in \Lambda$ are related to the $p$-adic $L$-functions $L_p^\alpha(f), L_p^\beta(f)$ of Amice–Vélu and Vishik in the following manner:
\[
(1.6) \quad \left( \begin{array}{c}
L_p^\alpha(f) \\
L_p^\beta(f)
\end{array} \right) = Q_{\alpha,\beta}^{-1} M_{\log} \cdot \left( \begin{array}{c}
L_p^\alpha(f) \\
L_p^\beta(f)
\end{array} \right),
\]
where $Q_{\alpha,\beta} = \left( \begin{array}{cc}
\alpha & \beta \\
\beta & -\alpha
\end{array} \right)$ and $M_{\log}$ is a certain “logarithm matrix”.

Project A. Show Expectation 1 for the signed $p$-adic $L$-functions. More precisely, for each
$\bullet \in \{\text{a, b}\}$, show that if $f \equiv g \pmod{p}$, then
\[
\mu(L_p^\bullet(f)) = 0 \implies \mu(L_p^\bullet(g)) = 0
\]
and the $\lambda$-invariants of $\Sigma$-imprimitive versions of $L_p^\bullet(f)$ and $L_p^\bullet(g)$ are equal.

Say $k = 2$ for simplicity. Similarly as in [GV00], the proof of this result would follow from the equality
\[
L_p^{\Sigma,\bullet}(f) \equiv u L_p^{\Sigma,\bullet}(g) \pmod{p\Lambda},
\]
for some unit $u \in \mathbb{Z}_p^\times$, which in turn would follow from establishing the congruence
\[
(1.7) \quad L_p^{\Sigma,\bullet}(f, \zeta - 1) \equiv u L_p^{\Sigma,\bullet}(g, \zeta - 1) \pmod{p\mathbb{Z}_p[\zeta]},
\]
for all $\zeta \in \mu_{p^\infty}$ and some $u \in \mathbb{Z}_p^\times$ independent of $\zeta$. However, a point of departure here from the $p$-ordinary setting is that (unless $a_p = b_p = 0$) the signed $p$-adic $L$-functions $L_p^\bullet(f), L_p^\bullet(g)$ are not directly related to twisted $L$-values, and so the arguments of [GV00, §3] do not suffice to cover this case. Nonetheless, it should be possible to exploit the result of [Vat99, Prop. 1.7], which amounts to the congruence
\[
L_p^{\Sigma,\bullet}(f, \zeta - 1) \equiv u L_p^{\Sigma,\bullet}(g, \zeta - 1) \pmod{p\mathbb{Z}_p[\zeta]},
\]
for both $\bullet \in \{\alpha, \beta\}$, together with (1.6) to establish (1.7). This will involve a detailed analysis of the values of $M_{\log}$ at $p$-power roots of unity, for which some of the calculations in [LLZ17] (see esp. loc.cit., Lem. 3.7]) might be useful.

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1Extending earlier work of Kobayashi, Pollack, Lei, and Sprung
Remark 1.1. The algebraic analogue of Project A has recently been established by Hatley–Lei (see [HL16, Thm. 4.6]). On the other hand, as shown in [LLZ11, Cor. 6.6], either of the main conjectures (1.5) is equivalent to Kato’s main conjecture (see [LLZ11, Conj. 6.2]). Thus from the discussion of §1.2 and the main result of [KKS17], we see that a successful completion of Project A would yield\(^2\) cases of the signed main conjectures beyond those covered by [Wan14] or [CÇSS17, Thm. B], where the following hypothesis is needed:  

there exists a prime \(\ell \neq p\) with \(\ell \mid N\) such that \(\bar{\rho}_f\) is ramified at \(\ell\).  

(cf. [KKS17, §1.2.3]).

1.4. On the anticyclotomic main conjecture of Bertolini–Darmon–Prasanna. Here we let \(F_{∞}/F\) be the anticyclotomic \(\mathbb{Z}_p\)-extension of an imaginary quadratic field \(K\) in which \(p\) splits, let \(f \in S_k(\Gamma_0(N))\), and let \(p \not| N\) be a prime. Assume also that every prime factor of \(N\) splits in \(K\); so \(K\) satisfies the Heegner hypothesis, and \(N^- = 1\) with the standard notation.

The Iwasawa–Greenberg main conjecture for the \(p\)-adic \(L\)-function \(L_p(f) \in \mathbb{Z}_p[[\text{Gal}(F_{∞}/F)]]\) introduced in [BDP13] predicts that

\[
\text{Char}_A(\text{Sel}_p(f)^{\vee})\Lambda_{\mathbb{Z}_p} \approx (L_p(f)),
\]

where \(\Lambda_{\mathbb{Z}_p} = \mathbb{Z}_p[[T]]\) and \(\text{Sel}_p(f)\) is a Selmer group defined by imposing local triviality (resp. no condition) at the primes above \(p\) (resp. \(p\)).

Project B. Show Expectation 1 for the \(p\)-adic \(L\)-functions of [BDP13]. That is, if \(f \equiv g \pmod{p}\), then \(\mu(L_p(f)) = \mu(L_p(g)) = 0\)\(^3\) and the \(\lambda\)-invariants of \(\Sigma\)-imprimitive versions of \(L_p(f)\) and \(L_p(g)\) are equal.

Similarly as for Project A, in weight \(k = 2\) this problem can be reduced to establishing the congruence

\[
L_p^\Sigma(f, \zeta - 1) \equiv uL_p^\Sigma(g, \zeta - 1) \pmod{p\mathbb{Z}_p[\zeta]}
\]

for all \(\zeta \in \mu_{p^\infty}\) and some \(u \in \mathbb{Z}_p^*\) independent of \(\zeta\). Now, by the \(p\)-adic Waldspurger formula of [BDP13, Thm. 5.13], the congruence of [KL16, Thm. 2.9] amounts to (1.9) for \(\zeta = 1\), and so a promising approach to Project B would be based on extending the result of [KL16, Thm. 2.9] to ramified characters.

Remark 1.2. When \(p\) is a good ordinary prime, the algebraic analogue of Project B has recently been established by Hatley–Lei (see [HL17, Prop. 4.2 and Thm. 5.4]). On the other hand, one can show that Howard’s divisibility towards Perrin-Riou’s Heegner point main conjecture implies one of the divisibilities predicted by (1.8) (see [How04, Thm. B] and [Cas17b, App. A]).

Similarly as in [KKS17], it should be possible to show (this is work in progress) that a suitable refinement of the Kolyvagin system arguments of [How04] combined with Wei Zhang’s proof of Kolyvagin’s conjecture [Zha14]\(^4\) yields the full equality (1.8). In particular, this would yield new cases of conjecture (1.8) with \(N^- = 1\) (not currently available in the literature), and even more cases (under a somewhat weaker version of Hypothesis ♠ in [Zha14], still with \(N^- = 1\)) after a successful completion of Project B.

Finally, in line with the previous remark, we note that the following should be possible:

Project C. Extend the results of [HL17] to the non-ordinary case.

\(^2\)Subject to the nonvanishing mod \(p\) of some “Kurihara number”

\(^3\)Note that in this case the vanishing of \(\mu\)-invariants is known under mild hypotheses by [Hsi14, Thm. B] and [Bur17, Thm. B]

\(^4\)Which can be seen as proving “primitivity” in the sense of [MR04] of the Heeger point Kolyvagin system
1.5. **On the $p$-part of the Birch–Swinnerton-Dyer formula for residually reducible primes.** Here we consider the primes $p > 2$ for which the associated residual representation $\bar{\rho}_f$ is reducible. For simplicity, assume that $f$ corresponds to an elliptic curve $E/\mathbb{Q}$ (admitting a rational $p$-isogeny with kernel $\Phi$). The combination of [GV00, Thm. 3.12] (with a key input from [Kat04, Thm. 17.4]) and [Gre99, Thm. 4.1] yields the $p$-part of the BSD formula for $E$ in analytic rank $0$, i.e., when $L(E,1) \neq 1$, provided the following holds:

\[
\text{(GV)} \quad \text{the } G_\mathbb{Q}\text{-action on } \Phi \subset E[p] \text{ is either} \quad \begin{cases} \text{ramified at } p \text{ and even, or} \\ \text{unramified at } p \text{ and odd.} \end{cases}
\]

Similarly as in the residually irreducible cases considered in [JSW17], the above result (applied to a suitable quadratic twist of $E$) would be an important ingredient in the following:

**Project D.** Prove the $p$-part of the BSD formula in analytic rank $1$ for elliptic curves $E$ and primes $p > 2$ for which (GV) does not hold.

Following the strategy of [JSW17] and [Cas17a], a key ingredient toward this would be the proof of the relevant cases of the anticyclotomic main conjecture ($1.8$). By the discussion in §1.2, this could be approached in the following steps:

1. establish the divisibility “$\supset$” in ($1.8$) (possibly after inverting $p$), based on a suitable refinement of the Kolyvagin system argument in [How04].
2. show that $\mu(L_p(f)) = 0$ based on the congruence of [Kri16, Thm. 3] between $L_p(f)$ and an anticyclotomic Katz $p$-adic $L$-function, and Hida’s results on the vanishing of $\mu$ for the latter.
3. letting $L_p^{\text{alg}}(f)$ be a generator of the characteristic ideal in ($1.8$), show that $\mu(L_p^{\text{alg}}(f)) = 0$ and $\lambda(L_p^{\text{alg}}(f)) = \lambda(L_p(f))$ based on an algebraic counterpart of [Kri16, Thm. 3] and the known cases of the main conjecture for the anticyclotomic Katz $p$-adic $L$-function.

After this is carried out, we could try to study the missing cases:

**Project E.** Prove the $p$-part of the BSD formula for elliptic curves $E/\mathbb{Q}$ at residually reducible primes $p > 2$ when:

- $L(E,1) \neq 0$ and (GV) doesn’t hold (complementing the cases that follow from [GV00]).
- $\text{ord}_{s=1} L(E,s) = 1$ and (GV) holds (complementing the cases covered by Project D).

Finally, we should note that $p = 2$ has been neglected throughout the above discussion, but one would of course like to understand this case as well. (See e.g. [CLZ17] for recent results in this direction.)

**References**


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5Note that there are other points where the residually irreducible hypothesis is used in [JSW17], e.g. in the “anticyclotomic control theorem” of [loc.cit., §3.3], but handling these should be relatively easy.


