COURSE AND PROJECT OUTLINE
MODULAR CURVES AND CYCLOTOMIC FIELDS
ARIZONA WINTER SCHOOL 2017

ROMYAR SHARIFI

COURSE OUTLINE

The prototypical example of an Iwasawa module is the inverse limit X of p-parts of class groups under norm maps up the tower of cyclotomic fields of p-power roots of unity. This group X is a finitely generated torsion module over the completed $\mathbb{Z}_p$-group ring of the tower of fields, known as an Iwasawa algebra. In this course, we study the structure of X.

Attached to the part of X on which complex conjugation acts by -1 is a characteristic ideal that provides a measure of the size of the module. Iwasawa conjectured that this ideal is generated by an element of the Iwasawa algebra determined the p-adic L-functions of even characters of the Galois group of the pth cyclotomic field over $\mathbb{Q}$. This statement, known as the Iwasawa main conjecture, provides a fundamental example of the links between algebraic and analytic objects in number theory. It was proven by Mazur and Wiles in 1984, building on ideas of Ribet in his 1976 proof of the converse to Herbrand's theorem.

Ribet and Mazur-Wiles look to the two-dimensional Galois representations attached to modular forms to construct unramified p-extensions of p-power cyclotomic fields. That is, the desired finite extensions are fixed by the kernel of a Galois representation ρ attached to a newform congruent to an Eisenstein series modulo a prime over p. The key idea is that ρ is residually reducible, and this enables one to find a 1-cocycle that gives rise to the unramified extension. More precisely, such cocycles can be used to construct a canonical homomorphism from X to the quotient of a space of cuspidal modular symbols of p-power levels by the action of an Eisenstein ideal of Hida's p-adic Hecke algebra.

There is a much more explicit map in the other direction, which is conjecturally inverse to the above map. At finite level, it takes modular symbols to cup products of cyclotomic p-units in Galois cohomology. All indications are that this is just one incarnation of a much more general phenomenon. The idea is that the geometry of locally symmetric spaces near their boundary strata describes the arithmetic of lower-dimensional objects.

Tentatively speaking, the structure of this course will be roughly as follows:

- background on Iwasawa modules and p-adic L-functions,
- the main conjecture and its proof,
- modular symbols and cup products,
- a refined conjecture, its consequences, and known results.

PROJECT OUTLINE

The projects will be of two sorts: purely Iwasawa theoretic and largely algebraic, and less Iwasawa theoretic but closely related to the above conjecture and its possible extensions. We give some ideas here of the sort of questions that could potentially form the bases of projects.

0.1. **Cohomology and Iwasawa modules.** Cup products of p-units provide information on the second stage of a certain augmentation filtration of an Iwasawa module over a Kummer extension of the field of p-power roots of unity. Are there instances where such cup products vanish where one can obtain deeper information on the structure of this Iwasawa module using higher operations (see [Sh1])? Can this be computed for a small irregular prime like 37, for instance, perhaps using a refinement of a method applied in [MS] to show the nonvanishing of a cup product? What other Galois groups and Iwasawa modules can one use these to describe the structure of more precisely (see for instance [Sh2])?

0.2. **The original conjecture.**

(1) For the conjecture of [Sh3], what is the arithmetic significance of a relation given by the choice of congruence subgroup, the cusps in the relative homology group, or a generator of the Eisenstein ideal for a Hecke operator dividing the level?
(2) What can be said about the relationship between other standard Iwasawa modules and modular symbols?
(3) Phrase a form of the conjecture relating K-groups of cyclotomic integer rings and higher weight modular symbols.
(4) Can one obtain a halfway decent bound on the p-rank of the class group of the pth cyclotomic field using modular symbols?

0.3. **Analogues over imaginary quadratic fields.** For an imaginary quadratic field F, one expects a similar relationship between modular symbols on a Bianchi space for a congurence subgroup of $\mathrm{GL}_2(\mathscr{O}_F)$ modulo Eisenstein and a second cohomology group of an ray class field of F (see [FKS]). Here, Manin symbols are replaced by symbols of Cremona (if F is Euclidean) and cyclotomic units are replaced by elliptic units.

(1) In what instances can one prove that there is a well-defined, canonical map in either direction?
(2) What if F is not Euclidean, but for instance has class number one?

REFERENCES

[FK] T. Fukaya, K. Kato, On conjectures of Sharifi, preprint, 2012, http://math.ucla.edu/~sharifi/sharificonj.pdf.

[FKS] T. Fukaya, K. Kato, R. Sharifi, Modular symbols in Iwasawa theory, *Iwasawa Theory 2012 - State of the Art and Recent Advances*, Contrib. Math. Comput. Sci. **7**, Springer, 2014, 177–219.

[Gr] R. Greenberg, Iwasawa theory – Past and Present, *Class field theory – its centenary and prospect (Tokyo, 1998)*, Adv. Stud. Pure Math. **30**, Math. Soc. Japan, Tokyo, 2001, 335–385.

[MS] W. McCallum, R. Sharifi, A cup product in the Galois cohomology of number fields, *Duke Math. J.* **120** (2003), 269–310.

[MW] B. Mazur, A. Wiles, Class fields of abelian extensions of $\mathbb{Q}$, *Invent. Math.* **76** (1984), 179–330.

[Oh] M. Ohta, Ordinary p-adic étale cohomology groups attached to towers of elliptic modular curves II, *Math. Ann.* **318** (2000), 557–583.

[Ri] K. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$, *Invent. Math.* **34** (1976), 151–162.

[Sh1] R. Sharifi, Massey products and ideal class groups, *J. reine angew. Math.* **603** (2007), 1–33.

[Sh2] R. Sharifi, On Galois groups of unramified pro-p extensions, *Math. Ann.* **342** (2008), 297–308.

[Sh3] R. Sharifi, A reciprocity map and the two variable p-adic L-function, *Annals of Math.* **173** (2011), 251–300.

[Sh4] R. Sharifi, Iwasawa theory (lecture notes), http://math.ucla.edu/∼iwasawa.pdf.

[Wa] P. Wake, Eisenstein Hecke algebras and conjectures in Iwasawa theory, *Algebra Number Theory* **9** (2015), 53–75.

[WW] P. Wake, C. Wang Erickson, Pseudo-modularity and Iwasawa theory, to appear in *Amer J. Math.*, arXiv:1505.05128v3.

The above is a sampling of relevant references. The lecture notes on Iwasawa theory [Sh4] may be the most useful for getting a head start is probably. The ideas of the paper of Ribet [Ri] are central to the course as well. The article [Gr] gives a fascinating survey of Iwasawa theory at the turn of the millenium. For the material later in the course, and an introduction to the papers [Sh3] and [FK], one might try reading the first part of the survey article [FKS].