

Euler systems (Arizona Winter School 2018 notes)

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Introduction

The theory of Euler systems is one of the most powerful tools available for studying the arithmetic of global Galois representations. However, constructing Euler systems is a difficult problem, and the list of known constructions was until recently accordingly rather short. In these lecture notes, we outline a general strategy for constructing new Euler systems in the cohomology of Shimura varieties: these Euler systems arise via pushforward of certain units on subvarieties.

We study in detail two special cases of this construction: the Euler system of Beilinson–Flach elements, where the underlying Shimura variety is the fibre product of two modular curves; and the Euler system of Lemma–Flach elements, arising in the cohomology of Siegel modular threefolds.

The lecture notes are structured as follows.

- In Chapter 1, we will review the definition of Euler systems for Galois representations, and their arithmetic application to the Bloch–Kato conjecture.
- In Chapter 2, we introduce some general tools for constructing global cohomology classes for Galois representations arising in geometry, assuming the existence of a supply of subvarieties of appropriate codimension and units on them. We also introduce Siegel units, which are the key for all the Euler system constructions to follow.
- Chapter 3 is largely motivational (and can be skipped at a first reading): it explains how one can use Rankin–Selberg-type integral formulas for L -functions as a guide to where to look for Euler systems.
- Chapter 4 is devoted to the construction of the Beilinson–Flach Euler system for pairs of modular forms of weight 2; and in Chapter 5, we discuss how to adapt this construction to pairs of modular forms of higher weight, using cohomology with coefficients.
- In Chapter 6 we explain the construction of the Lemma–Flach Euler system for genus 2 Siegel modular forms of parallel weight 3.
- In the concluding Chapter 7 we outline some projects.

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CHAPTER 1

Galois representations and Galois cohomology

References: for §§1.1—1.3, an excellent source is Bellaïche’s CMI notes on the Bloch–Kato conjecture.

1.1. Galois representations

1.1a. Definitions. Let K be a number field, \bar{K} its algebraic closure, $G_K = \text{Gal}(\bar{K}/K)$; and let p be a prime, and E a finite extension of \mathbf{Q}_p . We’re interested in representations of G_K on finite-dimensional E -vector spaces V .

We always assume that

- (1) $\rho : G_K \rightarrow \text{Aut}(V) \cong \text{GL}_d(E)$ is continuous (where $d = \dim(V)$), with respect to profinite topology of G_K and the p -adic topology on $\text{GL}_d(E)$.
- (2) V is “unramified almost everywhere”: for all but finitely many prime ideals v of K , we have $\rho(I_v) = \{1\}$, where I_v is an¹ inertia group at v .

1.1b. Examples.

The representation $\mathbf{Z}_p(1)$. Let $\mu_{p^n} = \{x \in \bar{K}^\times : x^{p^n} = 1\}$. Then μ_{p^n} is finite cyclic of order p^n and G_K acts on it.

The p -power map sends $\mu_{p^{n+1}} \rightarrow \mu_{p^n}$ and we define

$$\mathbf{Z}_p(1) := \varprojlim_n \mu_{p^n}, \quad \mathbf{Q}_p(1) := \mathbf{Z}_p(1) \otimes \mathbf{Q}_p.$$

This is a 1-dimensional continuous \mathbf{Q}_p -linear representation, unramified outside the primes dividing p ; G_K acts by “cyclotomic character” $\chi_{\text{cyc}} : G_K \rightarrow \mathbf{Z}_p^\times$.

(Notation: for any V , $n \in \mathbf{Z}$, we set $V(n) = V \otimes \mathbf{Q}_p(1)^{\otimes n}$.)

Tate modules of elliptic curves. A/K elliptic curve $\Rightarrow A(\bar{K})$ abelian group with G_K -action. Let $A(\bar{K})[p^n]$ subgroup of p^n -torsion points.

Define

$$T_p(A) := \varprojlim_n A(\bar{K})[p^n] \text{ (w.r.t. multiplication-by-} p \text{ maps)}, \quad V_p(A) := T_p(A) \otimes \mathbf{Q}_p.$$

This is a 2-dimensional continuous G_K -representation, unramified outside the set $\{v : v \mid p\} \cup \{v : A \text{ has bad reduction at } v\}$. (The same works for abelian varieties of any dimension g , giving $2g$ -dimensional representations of G_K .)

¹ I_v depends on a choice of prime of \bar{K} above v , but only up to conjugation in G_K , so whether or not V is unramified at v is well-defined.

Etale cohomology. Let X/K be a smooth algebraic variety. We can define vector spaces

$$H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p) \quad \text{for } 0 \leq i \leq 2 \dim X,$$

which are finite-dimensional p -adic Galois representations, unramified outside p and primes of bad reduction² of X .

1.1c. Representations coming from geometry. Our second example is a special case of the third: for an elliptic curve A , it turns out that we have $V_p(A) \cong H_{\text{ét}}^1(A_{\overline{K}}, \mathbf{Q}_p)(1)$.

DEFINITION. We say an E -linear Galois rep V comes from geometry if it is a subquotient of $H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p)(j) \otimes_{\mathbf{Q}_p} E$, for some variety X/K and some integers i, j .

So all my examples come from geometry. In these lectures we're only ever going to be interested in representations coming from geometry.

REMARK. Conjecturally the representations coming from geometry should be exactly those which are continuous, unramified almost everywhere, and *potentially semistable* at the primes above p (a technical condition from p -adic Hodge theory). This is called the **Fontaine–Mazur conjecture**. \diamond

1.2. L-functions of Galois representations

1.2a. Local Euler factors. Let V as above, v unramified prime. Then $\rho(\text{Frob}_v)$ is well-defined up to conjugacy, where Frob_v is the arithmetic Frobenius.

DEFINITION. The local Euler factor of V at v is the polynomial

$$P_v(V, t) := \det(1 - t \cdot \rho(\text{Frob}_v^{-1})) \in E[t].$$

Examples:

V	$P_v(V, t)$
\mathbf{Q}_p	$1 - t$
$\mathbf{Q}_p(n)$	$1 - \frac{t}{q_v^n}, \quad q_v = \#\mathbf{F}_v$
$H^1(A_{\overline{K}}, \mathbf{Q}_p)$	$1 - a_v(A)t + q_v t^2, \quad a_v(A) := 1 + q_v - \#A(\mathbf{F}_v)$

1.2b. Global L -functions (sketch). Assume V comes from geometry, and V is semisimple (direct sum of irreducibles). Then $P_v(V, t)$ has coefficients in $\overline{\mathbf{Q}}$ (Deligne); and there is a way of defining $P_v(V, t)$ for bad primes v (case $v \mid p$ is hardest).

Fix an embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Then we consider the product

$$L(V, s) := \prod_{v \text{ prime}} P_v(V, q_v^{-s})^{-1}.$$

Miraculously, this converges for $\Re(s) \gg 0$.

²This is a little delicate to define properly if we don't assume X to be proper over K . Formally, we say X has “good reduction” at v if it's isomorphic to the complement of a relative normal crossing divisor in a smooth proper $O_{K,v}$ -scheme.

E.g. for $V = \mathbf{Q}_p(n)$ this is $\zeta_K(s+n)$, where ζ_K is the Dedekind zeta function of K (which is just the Riemann zeta for $K = \mathbf{Q}$). For $V = H^1(A_{\overline{K}}, \mathbf{Q}_p)$, A an elliptic curve, it is the Hasse–Weil L -function $L(A/K, s)$.

CONJECTURE 1. *For V semisimple and coming from geometry, $L(V, s)$ has meromorphic continuation to $s \in \mathbf{C}$ with finitely many poles, and satisfies a functional equation relating $L(V, s)$ and $L(V^*, 1-s)$.*

Note that if V is semisimple and comes from geometry, the same is true³ of V^* , so the conjecture is well-posed. This conjecture is of course super-super-hard – the only cases where it is known is where we can relate V to something *automorphic*, e.g. a modular form.

There are lots of conjectures (and a rather smaller set of theorems) relating properties of arithmetic objects to values of their L -functions; the Birch–Swinnerton-Dyer conjecture is perhaps the best-known of these. As we’ve just seen, all the information about an elliptic curve you need to define its L -function is encoded in the Galois action on its Tate module; so can we express the BSD conjecture purely in terms of Galois representations? This will be the topic of the next section.⁴

1.3. Galois cohomology

1.3a. Setup. There is a cohomology theory for Galois representations⁵: for V an E -linear G_K -rep, we get E -vector spaces $H^i(K, V)$, zero unless $i = 0, 1, 2$. Mostly we care about H^0 and H^1 , which are given as follows

$$H^0(K, V) = V^{G_K}$$

$$H^1(K, V) = \frac{\{\text{cts fncs } s : G_K \rightarrow V \text{ such that } s(gh) = s(g) + gs(h)\}}{\{\text{fncs of the form } s(g) = gv - v \text{ for some } v \in V\}}.$$

These are well-behaved: short exact sequences of V ’s give long exact sequences of cohomology, for instance. Unfortunately they’re *not* finite-dimensional in general.

1.3b. The Kummer map. For $V = \mathbf{Q}_p(1)$ the Galois cohomology is related to the multiplicative group K^* . To see this, we have to first think a bit about cohomology with *finite* coefficients.

For any n , we have a short exact sequence

$$0 \longrightarrow \mu_{p^n} \longrightarrow \overline{K}^\times \xrightarrow{[p^n]} \overline{K}^\times \longrightarrow 0$$

which leads to a long exact sequence

$$0 \longrightarrow \mu_{p^n}^{G_K} \longrightarrow K^\times \xrightarrow{[p^n]} K^\times \longrightarrow H^1(K, \mu_{p^n})$$

³It is not obvious if this holds without the semisimplicity assumption.

⁴Actually the answer is “no, we can’t” – as far as I’m aware, there is no purely Galois-representation-theoretic statement that is precisely equivalent to BSD. But we can get pretty close, as we’ll shortly see.

⁵Technical point: our representations are all continuous, so we shall work with cohomology defined by continuous cochains, which is slightly different from the cohomology of G_K as an abstract group.

and thus an injection⁶

$$K^\times \otimes \mathbf{Z}/p^n \mathbf{Z} \hookrightarrow H^1(K, \mu_{p^n}).$$

Passing to the inverse limit we get a map (**Kummer map**)

$$\kappa_p : K^\times \otimes \mathbf{Z}_p \hookrightarrow H^1(K, \mathbf{Z}_p(1)) \quad \text{or} \quad K^\times \otimes \mathbf{Q}_p \hookrightarrow H^1(K, \mathbf{Q}_p(1)).$$

REMARK. This already shows that $H^1(K, \mathbf{Q}_p(1))$ can't be finite-dimensional, because K^\times has countably infinite rank. \diamond

The same argument works for elliptic curves: we get an embedding

$$E(K) \otimes \mathbf{Q}_p \hookrightarrow H^1(K, V_p(E)).$$

1.3c. Selmer groups. Since the groups $H^1(K, V)$ can be infinite-dimensional, it's useful to "cut down to size" by imposing extra conditions on our H^1 elements. We'll do this by localising at primes of K . Note that we have maps

$$H^i(K, V) \rightarrow H^i(K_v, V) \quad \text{for all primes } v,$$

and the local groups $H^i(K_v, V)$ are finite-dimensional.

DEFINITION. A local condition on V at prime v is an E -linear subspace $\mathcal{F}_v \subseteq H^1(K_v, V)$.

Examples:

- *strict* local condition $\mathcal{F}_{v,\text{strict}} = \{0\}$
- *relaxed* local condition $\mathcal{F}_{v,\text{relaxed}} = \text{everything}$
- *unramified* local condition

$$\mathcal{F}_{v,\text{ur}} = \text{image} (H^1(G_{K_v}/I_v, V^{I_v}) \rightarrow H^1(K_v, V))$$

- *Bloch–Kato “finite”* local condition $\mathcal{F}_{v,\text{BK}}$ (for $v \mid p$) – defined using p -adic Hodge theory.

DEFINITION. A Selmer structure is a collection $\mathcal{F} = (\mathcal{F}_v)_{v \text{ prime of } K}$, satisfying the following condition: for almost all v we have $\mathcal{F}_v = \mathcal{F}_{v,\text{ur}}$. If \mathcal{F} is a Selmer structure we define the corresponding Selmer group by

$$\text{Sel}_{\mathcal{F}}(K, V) = \{x \in H^1(K, V) : \text{loc}_v(x) \in \mathcal{F}_v \ \forall v\}.$$

THEOREM 1 (Tate). For any Selmer structure \mathcal{F} , the space $\text{Sel}_{\mathcal{F}}(K, V)$ is finite-dimensional over \mathbf{Q}_p .

SKETCH OF PROOF. It's easy to see that if this statement is true for one \mathcal{F} , it's true for any \mathcal{F} , since the *local* Galois cohomology groups $H^1(K_v, V)$ are all finite-dimensional. We now choose a particular Selmer structure \mathcal{F} (exercise: which?) such that $\text{Sel}_{\mathcal{F}}(K, V)$ is the image of the map

$$H^1(\text{Gal}(K^\Sigma/K), V) \hookrightarrow H^1(K, V),$$

where K^Σ is the maximal extension of K unramified outside some finite set of places Σ containing all infinite places, all places above p , and all places where V is ramified. This reduces us to what Tate actually proved, which is that the cohomology groups of $\text{Gal}(K^\Sigma/K)$ are finite-dimensional. \square

⁶In fact this is an isomorphism, because $H^1(K, \overline{K}^\times)$ is zero (“Hilbert’s theorem 90”)

We're mostly interested in three specific choices of Selmer structure, differing only in the choices of the \mathcal{F}_v at primes $v \mid p$: we define the *strict Selmer group* $\text{Sel}_{\text{strict}}(K, V)$ by taking $\mathcal{F}_v = \mathcal{F}_{v, \text{ur}}$ for $v \nmid p$, and $\mathcal{F}_v = \mathcal{F}_{v, \text{strict}}$ for $v \mid p$; and similarly the *relaxed Selmer group* and *Bloch–Kato Selmer group*.

Hence the strict, relaxed, and Bloch–Kato Selmer groups satisfy

$$\text{Sel}_{\text{strict}}(K, V) \subseteq \text{Sel}_{\text{BK}}(K, V) \subseteq \text{Sel}_{\text{relaxed}}(K, V).$$

REMARK. As will soon become clear, it is $\text{Sel}_{\text{BK}}(K, V)$ which is the most important of all. We care about $\text{Sel}_{\text{strict}}(K, V)$ and $\text{Sel}_{\text{relaxed}}(K, V)$ because they are easier to study, and will give us a stepping-stone towards $\text{Sel}_{\text{BK}}(K, V)$. \diamond

EXAMPLE. Recall that for $V = \mathbf{Q}_p(1)$ we had the Kummer map

$$K^\times \otimes \mathbf{Q}_p \hookrightarrow H^1(K, \mathbf{Q}_p(1)).$$

One can check that this induces isomorphisms

$$\begin{aligned} \mathcal{O}_K[1/p]^\times \otimes \mathbf{Q}_p &\xrightarrow{\cong} \text{Sel}_{\text{relaxed}}(K, \mathbf{Q}_p(1)), \\ \mathcal{O}_K^\times \otimes \mathbf{Q}_p &\xrightarrow{\cong} \text{Sel}_{\text{BK}}(K, \mathbf{Q}_p(1)). \end{aligned}$$

The strict Selmer group, on the other hand, should be zero; this is exactly Leopoldt's conjecture for K . \diamond

1.3d. The Bloch–Kato conjecture. Let V be a representation coming from geometry.

CONJECTURE 2 (Bloch–Kato). *We have*

$$\dim \text{Sel}_{\text{BK}}(K, V) - \dim H^0(K, V) = \text{ord}_{s=0} L(V^*(1), s).$$

There are refined versions using \mathbf{Z}_p -modules in place of \mathbf{Q}_p -vector spaces, which predict the leading term of the L -function up to a unit; but we won't go into these here.

Let's look at what the conjecture says in some example cases.

Example 1: $V = \mathbf{Q}_p$. Here $L(V^*(1), s) = L(\mathbf{Q}_p, s+1) = \zeta_K(s+1)$, so the right-hand side is the order of vanishing of $\zeta_K(s)$ at $s=1$, which is -1 for every K (there's a simple pole). The left-hand side is $\dim \text{Sel}_{\text{BK}}(K, \mathbf{Q}_p) - 1$, so the conjecture predicts that $\text{Sel}_{\text{BK}}(K, \mathbf{Q}_p) = 0$.

Exercise: Prove this. You'll need to use the finiteness of the ideal class group of K , together with the fact that for this representation the local condition $\mathcal{F}_{v, \text{BK}}$ agrees with $\mathcal{F}_{v, \text{ur}}$ for primes $v \mid p$.

Example 2: $V = \mathbf{Q}_p(1)$. Here $L(V^*(1), s) = \zeta_K(s)$. Inspecting the functional equation for Dedekind zeta functions, we see that $\text{ord}_{s=0} \zeta_K(s) = r_1 + r_2 - 1$, where r_1, r_2 are the numbers of real and complex places respectively. (In particular, if $K = \mathbf{Q}$, then $\zeta(0) = -\frac{1}{2}$ is finite and non-zero.) On the algebraic side, we have $H^0(K, \mathbf{Q}_p(1)) = 0$ and

$$\dim \text{Sel}_{\text{BK}}(K, V) = \dim_{\mathbf{Q}_p} (\mathcal{O}_K^\times \otimes \mathbf{Q}_p) = \text{rank } \mathcal{O}_K^\times.$$

So the Bloch–Kato conjecture here is exactly Dirichlet's unit theorem.

Example 3: Elliptic curves. If V is $V_p(E)$ for an elliptic curve E , then:

- the H^0 term is zero;
- the Kummer map lands inside the BK Selmer group, and gives an embedding

$$E(K) \otimes \mathbf{Q}_p \hookrightarrow \mathrm{Sel}_{\mathrm{BK}}(K, V),$$

so that $\dim \mathrm{Sel}_{\mathrm{BK}} \geq \mathrm{rank}(E/K)$, with equality iff the p -part of Sha is finite;

- $\mathrm{ord}_{s=0} L(V^*(1), s) = \mathrm{ord}_{s=1} L(E/K, s)$.

So this instance of Bloch–Kato is closely related to (but not quite the same as) the Birch–Swinnerton-Dyer conjecture.

REMARK. Notice that $L(V^*(1), s)$ is expected to be related to $L(V, -s)$ via a functional equation; but this functional equation will involve various Γ functions as factors, which can have poles, so the orders of vanishing of the two functions at 0 are not the same in general, as we saw for \mathbf{Q}_p and $\mathbf{Q}_p(1)$. On the Selmer-group side there’s a corresponding relation between $\mathrm{Sel}_{\mathrm{BK}}(K, V)$ and $\mathrm{Sel}_{\mathrm{BK}}(K, V^*(1))$ coming from the Poitou–Tate global duality theorem in Galois cohomology. One can check that these factors precisely cancel out: if $L(V, s)$ has a functional equation of the expected form, then the Bloch–Kato conjecture holds for $V^*(1)$ if and only if it holds for V . This is a wonderful (but rather involved) exercise. \diamond

1.4. Euler systems

We’ll now introduce the key subject of these lectures: Euler systems, which are tools for studying and controlling Selmer groups. In this section we’ll give the abstract definition of an Euler system, and explain (without proofs) why the existence of an Euler system for some Galois representation has powerful consequences for Selmer groups.

References: The standard work on this topic is Karl Rubin’s orange book *Euler Systems* [Rub00]. There are also two alternative accounts in Rubin’s 2004 Park City lecture notes, and in the book *Kolyvagin Systems* [MR04] by Mazur and Rubin.

1.4a. The definition. Let:

- V a $G_{\mathbf{Q}}$ -representation (for simplicity)
- $T \subset V$ a $G_{\mathbf{Q}}$ -stable \mathbf{Z}_p -lattice
- Σ a finite set of primes containing p and all ramified primes for V

Since V is a $G_{\mathbf{Q}}$ -rep, we can consider it as a G_K -rep for any number field K and form $H^i(K, V)$, and there are **corestriction** or **norm** maps

$$\mathrm{norm}_K^L : H^i(L, V) \rightarrow H^i(K, V) \quad \text{if } L \supset K.$$

If K is Galois, $H^i(K, V)$ is a module over $\mathbf{Q}_p[\mathrm{Gal}(K/\mathbf{Q})]$. Similarly for cohomology of lattices $H^i(K, T)$.

DEFINITION. An Euler system for (T, Σ) is a collection $\mathbf{c} = (c_m)_{m \geq 1}$, where $c_m \in H^1(\mathbf{Q}(\mu_m), T)$, satisfying the following compatibility for any $m \geq 1$ and ℓ prime:

$$\text{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{m\ell})}(c_{m\ell}) = \begin{cases} c_m & \text{if } \ell \in \Sigma \text{ or } \ell \mid m \\ P_\ell(V^*(1), \sigma_\ell^{-1}) \cdot c_m & \text{otherwise} \end{cases}$$

where σ_ℓ is the image of Frob_ℓ in $\text{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$. An Euler system for V is an Euler system for (T, Σ) , for some $T \subset V$ and some Σ .

This definition is not very transparent, I admit! Fear not: we'll see an example before too long. Intuitively, each class c_m has “something to do with” the L -function $L(V^*(1), s)$ with its Euler factors at primes dividing $m\Sigma$ missing⁷; so when we compare elements for different m , the Euler factors appear.

The main reason to care about these objects is the following theorem, which is due to Rubin [Rub00], building on earlier work of Kato [Kat04], Kolyvagin [Kol91], and Thaine [Tha88]:

THEOREM 2. Suppose \mathbf{c} is an Euler system for (T, Σ) with c_1 non-zero, and suppose V satisfies various technical conditions. Then $\text{Sel}_{\text{strict}}(\mathbf{Q}, V^*(1))$ is zero.

For the purposes of these lectures we don't need to know how this theorem is proved – our goal is to understand how to *build* Euler systems, which is a separate problem. If you do want to know about the proof, then see the references listed above.

REMARK.

- The technical conditions are to do with the image of $G_{\mathbf{Q}}$ in $\text{GL}(V)$. This needs to be “large enough” in a certain precise sense, which in particular implies that V is irreducible.
- For the proof of the theorem, we don't actually need c_m to be defined for all m ; it's enough to have c_m for all integers m of the form $p^k m_0$, where $k \geq 0$ and m_0 is a square-free product of primes not in Σ .
- More generally, one can also define Euler systems for G_K -representations, for K a number field. In place of cyclotomic fields, one has to have classes over different ray class fields of K . However, we'll only work with $K = \mathbf{Q}$ here.
- There is also a notion of “anticyclotomic Euler system”, which applies when you have a representation V of G_K , a quadratic extension L/K , and cohomology classes for V over the *anticyclotomic extensions* of L , which are the abelian extensions of L such that conjugation by $\text{Gal}(L/K)$ acts on their Galois groups by -1 . The most important example of an anticyclotomic Euler system is Kolyvagin's **Euler system of Heegner points** [Kol91], where $K = \mathbf{Q}$, $V = V_p(E)$ for E an elliptic curve, and L is an imaginary quadratic field. Other examples of anticyclotomic Euler systems have recently been found by Cornut, and by Jetchev and his collaborators.

◇

⁷This becomes more precise if you work with the *equivariant* L -function $L(V^*(1), \mathbf{Q}(\mu_m)/\mathbf{Q}, s)$ which is a Dirichlet series taking values in the group ring $\mathbf{C}[(\mathbf{Z}/m\mathbf{Z})^\times]$ rather than just in \mathbf{C} , encoding the L -values of V twisted by Dirichlet characters modulo m . The definition of this function only makes sense if you drop the Euler factors at primes dividing m .

1.4b. Cyclotomic units. We're going to build an Euler system for $V = \mathbf{Q}_p(1)$. Recall that we have Kummer maps $K^\times \hookrightarrow H^1(K, \mathbf{Z}_p(1))$. Also, for L/K finite, we have a commutative square

$$\begin{array}{ccc} L^\times & \xrightarrow{\kappa_p} & H^1(L, \mathbf{Z}_p(1)) \\ \text{norm}_K^L \downarrow & & \downarrow \text{norm}_K^L \\ K^\times & \xrightarrow{\kappa_p} & H^1(K, \mathbf{Z}_p(1)) \end{array}$$

where the left-hand norm map is the usual field norm, and the right-hand one is the Galois corestriction. So we have to find good elements of the multiplicative groups of cyclotomic fields, satisfying compatibilities under the norm maps.

Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}^\times$ and let $\zeta_m = \iota^{-1}(e^{2\pi i/m}) \in \mu_m$.

DEFINITION. For $m > 1$, set $u_m = 1 - \zeta_m \in \mathbf{Q}(\mu_m)^\times$.

A pleasant computation (exercise!) shows that⁸

$$\text{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{m\ell})} u_m = \begin{cases} u_m & \text{if } \ell \mid m \\ (1 - \sigma_\ell^{-1}) \cdot u_m & \text{if } \ell \nmid m \text{ and } m > 1 \\ \ell & \text{if } m = 1 \end{cases}$$

This is almost what we need for an Euler system, but there are two problems: firstly, there is no sensible way to define u_1 ; secondly, we are seeing Euler factors at *all* primes, whereas we only want to see them for primes outside Σ (and Σ can't be empty because it has to contain p). We can get around both of these problems by setting

$$v_m = \begin{cases} u_m & \text{if } p \mid m, \\ \text{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{pm})}(u_{pm}) & \text{if } p \nmid m \text{ (including } m = 1). \end{cases}$$

THEOREM 3. The classes $c_m = \kappa_p(v_m)$ are an Euler system for $(\mathbf{Z}_p(1), \{p\})$. \square

1.4c. Soulé twists. Rubin's theorem applied directly to the cyclotomic unit Euler system isn't actually very interesting (it follows easily from class field theory that $\text{Sel}_{\text{strict}}(\mathbf{Q}, \mathbf{Q}_p) = 0$). However, there is a notion of *twisting* for Euler systems.

THEOREM 4. Let $\chi : G_{\mathbf{Q}} \rightarrow \mathbf{Z}_p^\times$ be a continuous character unramified outside Σ (e.g. any power of the cyclotomic character). Then there is a canonical bijection $\mathbf{c} \mapsto \mathbf{c}^\chi$ between Euler systems for T and for the twist $T(\chi)$.

Note that the ‘‘bottom class’’ c_1^χ in the twisted Euler system depends on whole system $\mathbf{c} = \{c_m\}_{m \geq 1}$, not just on c_1 . So even if $c_1 \neq 0$ we might have $c_1^\chi = 0$, and we have to check carefully that the twisted Euler system satisfies the conditions for Rubin's theorem.

The twists of the cyclotomic unit Euler system have many applications in number theory; see e.g. §3.2 of [Rub00]. For instance, they play a major role in Huber

⁸This is stated in Rubin's book ‘‘Euler Systems’’, §3.2, but with a sign error: he sets $u_m = \zeta_m - 1$, which doesn't quite work, since $\text{norm}_{\mathbf{Q}(\mu_4)}^{\mathbf{Q}(\mu_8)}(\zeta_8 - 1) \neq \zeta_4 - 1$.

and Kings' proof of the Bloch–Kato conjecture for $\mathbf{Q}_p(n)$ for all $n \in \mathbf{Z}$, an account of which can be found in [CRSS15].

A toolkit for building Euler systems

2.1. Étale cohomology and the Hochschild–Serre spectral sequence

(References: not as many as there should be. Jannsen’s article “Continuous étale cohomology” [Jan88] has the details, but it is not an easy read.)

We saw before that, for a variety X/K , the étale cohomology groups $H_{\text{ét}}^i(X_{\overline{K}}, \mathbf{Q}_p)$ were an interesting source of Galois representations.

But this isn’t the only thing we can do with étale cohomology. Rather than base-extending to \overline{K} , we can also take étale cohomology of X/K directly¹; there are groups $H_{\text{ét}}^i(X, \mathbf{Q}_p(m))$ for all i and m . These “absolute” étale cohomology groups are *not* themselves Galois representations, but it turns out that these are related to the Galois cohomology of the étale cohomology over \overline{K} :

THEOREM 5 (Jannsen). *For any variety X/K , and any n , there is a convergent “Hochschild–Serre” spectral sequence*

$$E_2^{ij} = H^i\left(K, H_{\text{ét}}^j(X_{\overline{K}}, \mathbf{Q}_p)(n)\right) \Rightarrow H_{\text{ét}}^{i+j}(X, \mathbf{Q}_p(n)).$$

In particular, we get edge maps $H^i(X, \mathbf{Q}_p(n)) \rightarrow H^i(X_{\overline{K}}, \mathbf{Q}_p(n))^{G_K}$, and if $F^1 H^i$ denotes the kernel of this map (the “homologically trivial” classes), there is a map

$$F^1 H^i(X, \mathbf{Q}_p(n)) \rightarrow H^1\left(K, H^{i-1}(X_{\overline{K}}, \mathbf{Q}_p)(n)\right).$$

So, if X is defined over \mathbf{Q} and V is the Galois representation $H^{i-1}(X_{\overline{\mathbf{Q}}})$ (or a direct summand of it), we can try to construct an Euler system for V by building classes in $F^1 H^i(X_{\mathbf{Q}(\mu_m)})$ for varying m .

How will we do this? We’ll use geometry! To be precise, we’ll rely on the following rather simple bag of tricks:

- **Cup products:** étale cohomology has cup-product maps

$$H^i(X, \mathbf{Q}_p(m)) \times H^j(X, \mathbf{Q}_p(n)) \rightarrow H^{i+j}(X, \mathbf{Q}_p(m+n)).$$

- **Kummer maps:** if $f \in \mathcal{O}(X)^\times$ is a unit in the ring of rational functions on X , then there is a class $\kappa_p(f) \in H^1(X, \mathbf{Q}_p(1))$.
- **Pushforward maps:** if $Z \subset X$ is a closed subvariety of codimension d (and X and Z are both smooth), then there are pushforward maps

$$H^i(Z, \mathbf{Q}_p(n)) \rightarrow H^{i+2d}(X, \mathbf{Q}_p(n+d)).$$

¹Technical point: what we actually want here is “continuous étale cohomology” in the sense of Jannsen. This is consistent with our use of continuous cochains to define cohomology of Galois representations.

In particular, the pushforward of the identity class $1_Z \in H^0(Z, \mathbf{Q}_p(0))$ is a class in $H^{2d}(X, \mathbf{Q}_p(d))$, the *cycle class* of Z .

So if we have a good supply of units on X , or of subvarieties of X (or of subvarieties of X with units on them, etc) then we have some objects to play with; and we can try to write down classes landing in the “right” cohomological degree to map into H^1 of our target Galois representation.

If you have a random variety, it’s not clear how to find lots of subvarieties, or lots of units, on it; but we’re going to home in on the case where X is a *Shimura variety* – a variety coming from automorphic theory, such as a modular curve. Then we can try and write down units and subvarieties using automorphic ideas.

2.2. Modular curves and modular forms

(References: Diamond–Shurman [DS05], Darmon–Diamond–Taylor [DDT97].)

We’re particularly interested in the Galois representations associated to modular forms, which come from geometry via modular curves. We’ll consider weight 2 modular forms first, as these are the simplest to handle.

2.2a. Modular curves. For $N \geq 1$ let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : c = 0, d = 1 \pmod{N} \right\}.$$

This acts on the upper half-plane \mathcal{H} via $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. It turns out that the quotient is naturally an algebraic variety:

THEOREM 6. *For $N \geq 4$ there is an algebraic variety $Y_1(N)$ over \mathbf{Q} with the following properties:*

- $Y_1(N)$ is a smooth geometrically connected affine curve.
- For any field extension² F/\mathbf{Q} , the F -points of $Y_1(N)$ biject with isomorphism classes of pairs (E, P) , where E/F is an elliptic curve and $P \in E(F)$ is a point of order N on E .
- $Y_1(N)(\mathbf{C}) \cong \Gamma_1(N) \backslash \mathcal{H}$, via the map sending $\tau \in \mathcal{H}$ to (E_τ, P_τ) where $E_\tau = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\tau)$ and $P_\tau = 1/N \pmod{\mathbf{Z} + \mathbf{Z}\tau}$.

(Much stronger theorems are known – for instance, $Y_1(N)$ has a canonical smooth model over $\mathbf{Z}[1/N]$ – but we won’t need this just now.)

REMARK. There are two different choices of conventions for \mathbf{Q} -models for $Y_1(N)$; everyone agrees what $Y_1(N)$ means over \mathbf{C} , but there are two different ways to descend it to \mathbf{Q} , classifying elliptic curves with either a point of order N (our convention) or an embedding of the group scheme μ_N (the alternative convention). \diamond

²Any \mathbf{Q} -algebra, in fact; this is important if you want to make precise the idea that $Y_1(N)$ represents a functor.

2.2b. Galois representations. We can use these rational models of modular curves to attach Galois representations to modular forms. Let $f = \sum a_n q^n$ be a cuspidal modular eigenform of weight 2 and level $\Gamma_1(N)$, normalised so that $a_1 = 1$. By a theorem of Shimura, there is a number field L such that all $a_n \in L$. We shall fix an embedding $\iota : L \hookrightarrow \overline{\mathbf{Q}}_p$, and assume that our p -adic coefficient field E/\mathbf{Q}_p contains the image of ι .

DEFINITION. We let $V_p(f)$ be the largest subspace of $H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes E$ on which the Hecke operators $T(\ell)$, for $\ell \nmid N$, act as multiplication by $a_\ell(f)$.

By construction, $V_p(f)$ is an E -linear Galois representation coming from geometry. However, one can also show that

- (1) $V_p(f)$ is 2-dimensional and irreducible.
- (2) $V_p(f)$ is a direct summand of $H_{\text{ét}}^1$ (not just a subspace).
- (3) For $\ell \nmid pN$, $V_p(f)$ is unramified at ℓ and the trace of Frob_ℓ^{-1} on $V_p(f)$ is $a_\ell(f)$. More precisely, the local Euler factor is given by

$$P_\ell(V_p(f), t) = 1 - a_\ell(f)t + \ell\chi(\ell)t^2,$$

where χ is the character of f .

- (4) $V_p(f)^* = V_p(f \otimes \chi^{-1})(1)$.

It follows from (3) that (up to finitely many bad Euler factors at primes $\ell \mid pN$)³ the global L -series $L(V_p(f), s)$ is just the L -series of f ,

$$L(f, s) = \sum a_n(f)n^{-s}.$$

In particular if $L = \mathbf{Q}$, so that f corresponds to an elliptic curve A , then we have $V_p(f) \cong H_{\text{ét}}^1(A_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \cong V_p(A)(-1)$.

REMARK. Warning: in Diamond–Shurman chapter 9, the representation they denote by $\rho_{f,p}$ is the *dual* of our $V_p(f)$, which is compensated for by the fact that they use arithmetic Frobenius Frob_p rather than geometric Frobenius Frob_p^{-1} to define the Euler factor. The same applies to Romyar Sharifi’s notes at this Arizona Winter School: the representation (ρ_f, V_f) defined in §3.5 of his notes is the dual of our $V_p(f)$. \diamond

2.2c. Tensor products. Later on, we’ll be interested in *tensor products* of Galois representations associated to modular forms. If you take two newforms f, g (both with coefficients in E) and let V be the four-dimensional Galois representation $V = V_p(f) \otimes V_p(g)$, then using the Kunneth formula for étale cohomology you can show that V is a direct summand of $H_{\text{ét}}^2(Y_1(N)_{\overline{\mathbf{Q}}}^2, \mathbf{Q}_p) \otimes E$, for any N divisible by N_f and N_g .

The L -function attached to this tensor product representation is a rather classical object: it’s the so-called *Rankin–Selberg convolution L -function* of f and g , denoted

³In fact, if f is a newform, then $L(f, s)$ and $L(V_p(f), s)$ have the same Euler factors at the bad primes too, although this is much harder to check. This doesn’t work for the Rankin–Selberg L -function; the “naive” Rankin–Selberg L -series (\ddagger) frequently has the wrong local factors at the bad primes, even if f and g are newforms.

by $L(f \otimes g, s)$. Up to finitely many bad Euler factors, this agrees with the Dirichlet series

$$(\ddagger) \quad L(\chi_f \chi_g, 2s - 2) \sum_{n \geq 1} a_n(f) a_n(g) n^{-s}.$$

2.3. Numerology

For instance, let's suppose we want to build an Euler system for $V_p(f)$, where f is a modular form of weight 2. Since we can twist Euler systems, we can choose to work with $V_p(f)(n)$ for any integer n .

Because $Y = Y_1(N)_{\mathbf{Q}}$ is affine, we have $H_{\text{ét}}^2(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) = 0$, and $H_{\text{ét}}^1(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(n)$ contains $V_p(f)(n)$ as a direct summand. So the Hochschild–Serre spectral sequence gives us a map

$$H_{\text{ét}}^2(Y, \mathbf{Q}_p(n)) \rightarrow H^1(\mathbf{Q}, H_{\text{ét}}^1(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p)(n)) \rightarrow H^1(\mathbf{Q}, V_p(f)(n)).$$

How can we get at the groups $H_{\text{ét}}^2(Y, \mathbf{Q}_p(n))$ using our geometric toolkit?

- For $n \leq 0$ this is hopeless, because our toolkit will only ever give classes in $H^i(-, \mathbf{Q}_p(n))$ for $n \geq \frac{i}{2}$ (check this!)
- For $n = 1$, you can use cycle classes of codimension 1 subvarieties of Y – i.e., points. This is Kolyvagin's original approach [Kol91]: to build an Euler system using cycle classes of Heegner points. However, this gives an anticyclotomic Euler system (relative to some choice of imaginary quadratic field), not a full Euler system in the sense of §1.4a.⁴
- For $n = 2$, you can use cup-products of units: the Kummer map gives you classes in $H_{\text{ét}}^1(Y, \mathbf{Q}_p(1))$, and the cup-product of two such classes lands in $H_{\text{ét}}^2(Y, \mathbf{Q}_p(2))$. This is Kato's approach [Kat04].
- $n \geq 3$ can also be made to work similarly (but gives no more information than for $n = 2$).

We can also ask the same question for $V_p(f) \otimes V_p(g)$, using the geometry of $Y \times Y$. Again, different twists n give very different geometric setups; and taking n too small is hopeless – you want $n \geq 2$ at least. The sensible choices are:

- $n = 3$: we can get classes here as cup-products $\kappa_p(f_1) \cup \kappa_p(f_2) \cup \kappa_p(f_3)$, where f_1, f_2, f_3 are units on $Y \times Y$.
- $n = 2$: we can get classes by taking a curve $Z \subset Y \times Y$ and a unit $f \in \mathcal{O}(Z)^\times$, and pushing forward $\kappa_p(f) \in H_{\text{ét}}^1(Z, \mathbf{Q}_p(1))$ along the embedding $Z \hookrightarrow Y \times Y$.

The $n = 3$ approach has, I believe, never been carried out (and people have tried very hard to make it work without success). The $n = 2$ approach leads to the Euler system of Beilinson–Flach elements, which we'll discuss later in these lectures.

⁴This is an instance of a general phenomenon. We have seen that one needs $i \leq 2n$ for geometric techniques to work. It turns out that in the boundary case $i = 2n$, one can only work with cycle classes of subvarieties (not with units); and these cannot give a full Euler system, only an anticyclotomic one.

2.4. Changing the field and changing the level

To build an Euler system using the Hochschild–Serre spectral sequence, we need to build classes in $H^i(X_K, \mathbf{Q}_p(n))$ as K varies over cyclotomic fields. It turns out that, for modular curves, we can “sneak up” on this field extension by varying the level of our modular curves instead.

DEFINITION. We write μ_m° for the \mathbf{Q} -variety of primitive m -th roots of unity.

Concretely, this is the 0-dimensional subvariety of the affine line cut out by $\Phi_m(X) = 0$, where Φ_m is the m -th cyclotomic polynomial. This variety is connected (since the cyclotomic polynomials are irreducible over \mathbf{Q}) but not, of course, geometrically connected once $m > 2$.

Hence, for any variety X/\mathbf{Q} , we can consider the product variety $X \times \mu_m^\circ$, which is also a variety over \mathbf{Q} .

PROPOSITION 1. For any i, m, n , we have isomorphisms of $G_{\mathbf{Q}}$ -representations

$$H_{\text{ét}}^i\left((X \times \mu_m^\circ)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p\right) \cong \text{Ind}_{G_{\mathbf{Q}(\mu_m)}}^{G_{\mathbf{Q}}} H_{\text{ét}}^i\left(X_{\overline{\mathbf{Q}}}, \mathbf{Q}_p\right)$$

and isomorphisms of \mathbf{Q}_p -vector spaces

$$H_{\text{ét}}^i\left(X_{\mathbf{Q}(\mu_m)}, \mathbf{Q}_p(n)\right) \cong H_{\text{ét}}^i\left(X \times \mu_m^\circ, \mathbf{Q}_p(n)\right).$$

(This is a form of Shapiro’s lemma; it corresponds to the fact that $\mu_m^\circ = \text{Spec } \mathbf{Q}(\mu_m)$, and hence $X_{\mathbf{Q}} \times \mu_m^\circ$ is the image of $X_{\mathbf{Q}(\mu_m)}$ under the forgetful functor from $\mathbf{Q}(\mu_m)$ -varieties to \mathbf{Q} -varieties.)

This is useful to us because, if $X = Y_1(N)$, the base-extension $Y_1(N) \times \mu_m^\circ$ is *also* a modular curve. More precisely, for any open compact subgroup $U \subset \text{GL}_2(\mathbf{A}_f)$, there is an algebraic curve $Y(U)$ defined over \mathbf{Q} , whose \mathbf{C} -points are the quotient

$$(1) \quad Y(U)(\mathbf{C}) = \text{GL}_2^+(\mathbf{Q}) \backslash [\mathcal{H} \times \text{GL}_2(\mathbf{A}_f)/K].$$

If U is the subgroup

$$U_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\widehat{\mathbf{Z}}) : c = 0, d = 1 \pmod{N\widehat{\mathbf{Z}}} \right\}$$

then $Y(U)$ is just $Y_1(N)$. However, if we set $U' = \{u \in U_1(N) : \det(u) = 1 \pmod{m}\}$, then $Y(U')$ is canonically isomorphic to $Y_1(N) \times \mu_m^\circ$, and the action of the quotient U/U' on $Y(U')$ matches up with the Galois action via the usual isomorphism $\text{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q}) \cong (\mathbf{Z}/m\mathbf{Z})^*$.

This transports our problem – constructing cohomology classes for $Y_1(N)$ over varying cyclotomic fields – into a more “automorphic” problem: constructing cohomology classes for modular curves over \mathbf{Q} of varying levels.

REMARK. To some extent this is just a superficial change of language. However, it seems to be a helpful one, as will be clear from our proofs of norm relations later in these lectures. \diamond

2.5. Siegel units

As we saw above, we can get potentially useful cohomology classes if we have a source of units in the coordinate rings of our varieties. Fortunately, for modular curves, we have lots of nice units at our disposal. (References: §§1–2 of [Kat04] are the definitive source; [Lan87] is also useful.)

2.5a. The construction. Let U be an open compact subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$ (such as the group $U_1(N)$ from the previous section).

DEFINITION. A **modular unit** of level U is a unit in the coordinate ring of the algebraic variety $Y(U)$.

This definition is very clean, but hard to work with concretely. So we'll unwrap it a bit. Recall that $Y(U)(\mathbf{C})$ is defined as a quotient of $\mathcal{H} \times \mathrm{GL}_2(\mathbf{A}_f)$, so the image of $\mathcal{H} \times \{1\}$ in this quotient is a connected component of $Y(U)(\mathbf{C})$. It turns out that this image is exactly $\Gamma \backslash \mathcal{H}$, where Γ is the discrete group $U \cap \mathrm{GL}_2^+(\mathbf{Q})$ (which is commensurable with $\mathrm{SL}_2(\mathbf{Z})$). So we get a map

$$\left(\begin{array}{c} \text{modular units} \\ \text{of level } U \end{array} \right) \longrightarrow \left(\begin{array}{c} \text{nowhere-zero holomorphic fncs} \\ \text{on } \Gamma \backslash \mathcal{H} \text{ with finite-order poles at cusps} \end{array} \right).$$

Fact: This map is *injective*, because the Galois group acts transitively on the components of $Y(U)$. \square

For a general subgroup U the image is a little fiddly to describe. However, for some nice subgroups we can make it very concrete:

PROPOSITION 2. Let $U(N) \subset \mathrm{GL}_2(\widehat{\mathbf{Z}})$ be the kernel of the reduction map $\mathrm{GL}_2(\widehat{\mathbf{Z}}) \rightarrow \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$, and $\Gamma(N) = U(N) \cap \mathrm{SL}_2(\mathbf{Z})$. Then the modular units of level $U(N)$ are precisely the functions on $\Gamma(N) \backslash \mathcal{H}$ which are holomorphic and nonzero away from the cusps, are meromorphic at the cusps, and have q -expansion coefficients in $\mathbf{Q}(\mu_N)$. \square

We're going to construct some "special" modular units of level $U(N)$, using nothing but classical 19th-century elliptic function theory. These functions are called **Siegel units** and they are really amazingly powerful gadgets. In fact, you can recover virtually every known example of an Euler system by starting from Siegel units!

DEFINITION. Let $\alpha, \beta \in \mathbf{Q}/\mathbf{Z}$, not both zero. Define the function $g_{\alpha, \beta} : \mathcal{H} \rightarrow \mathbf{C}$ as follows: write $(\alpha, \beta) = (a/N, b/N)$ for some $N \geq 1$ and $a, b \in \mathbf{Z}$, with $0 \leq a < N$ without loss of generality. Then

$$g_{\alpha, \beta}(\tau) = q^w \prod_{n \geq 0} \left(1 - q^{n+a/N} \zeta_N^b\right) \prod_{n \geq 1} \left(1 - q^{n-a/N} \zeta_N^{-b}\right),$$

where $q = e^{2\pi i \tau}$ and $w = \frac{1}{12} - \frac{a}{N} + \frac{a^2}{2N^2}$.

This is well-defined (independent of the choice of common denominator N). We'd like to say it's modular of level N , but this doesn't quite work: acting on it by an element of $\Gamma(N)$ multiplies it by a root of unity. These error terms can be killed by a very simple modification:

DEFINITION (Siegel units). For $c > 1$ coprime to 6 and to the order of α, β in \mathbf{Q}/\mathbf{Z} , let

$$c\mathcal{G}_{\alpha,\beta} = \frac{(g_{\alpha,\beta})^{c^2}}{g_{c\alpha,c\beta}}.$$

PROPOSITION 3. The functions $c\mathcal{G}_{\alpha,\beta}$, for $(\alpha, \beta) \in (\frac{1}{N}\mathbf{Z}/\mathbf{Z})^{\oplus 2} - \{(0,0)\}$, are modular units of level $U(N)$. The left action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ on $Y(U(N))$ transforms these units via the rule

$$c\mathcal{G}_{\alpha,\beta} \mid \sigma = c\mathcal{G}_{\alpha',\beta'}, \quad \text{where } (\alpha', \beta') = (\alpha, \beta)\sigma. \quad \square$$

In particular, because $(0, \frac{1}{N})$ is preserved by right-multiplication by matrices of the form $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, which give the action of the quotient $U_1(N)/U(N)$, we see that:

PROPOSITION 4. The function $c\mathcal{G}_{0,1/N}$ is a modular unit of level $U_1(N)$. □

2.5b. Changing the level: the basic norm relation.

THEOREM 7. Let $\alpha, \beta \in \mathbf{Q}/\mathbf{Z}$, not both zero, and let $A \geq 1$. Then we have the three relations

$$(2) \quad \prod_{\alpha': A\alpha'=\alpha} c\mathcal{G}_{\alpha',\beta}(\tau) = c\mathcal{G}_{\alpha,\beta}(A^{-1}\tau),$$

$$(3) \quad \prod_{\beta': A\beta'=\beta} c\mathcal{G}_{\alpha,\beta'}(\tau) = c\mathcal{G}_{\alpha,\beta}(A\tau),$$

$$(4) \quad \prod_{\substack{\alpha',\beta' \\ A(\alpha',\beta')=(\alpha,\beta)}} c\mathcal{G}_{\alpha',\beta'}(\tau) = c\mathcal{G}_{\alpha,\beta}(\tau).$$

SKETCH OF PROOF. Note that (1) and (2) imply (3), and (2) follows from (1) via the action of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; so it suffices to prove (1). This can be bashed out directly from the infinite product formula, but there is a much slicker argument in [Kat04], involving a 2-variable theta function $c\theta(\tau, z)$ such that $c\theta(\tau, \alpha\tau + \beta) = c\mathcal{G}_{\alpha,\beta} \cdot c\theta(\tau, z)$. □

The most important relation is (3), which can be written in a more conceptual way using push-forward maps between modular curves. Suppose ℓ is a prime; then there's a quotient map $\pi : Y_1(N\ell) \rightarrow Y_1(N)$, and associated to this is a norm map $\pi_* : \mathcal{O}(Y_1(N\ell))^\times \rightarrow \mathcal{O}(Y_1(N))^\times$, characterised by

$$(\pi_* f)(x) = \prod_{y \in \pi^{-1}(x)} f(y) \quad \text{for } x \in \Gamma_1(N) \backslash \mathcal{H}.$$

COROLLARY 1. The Siegel units satisfy

$$\pi_*(c\mathcal{G}_{0,1/N\ell}) = \begin{cases} c\mathcal{G}_{0,1/N} & \text{if } \ell \mid N, \\ c\mathcal{G}_{0,1/N} \cdot (c\mathcal{G}_{0,u/N})^{-1} & \text{if } \ell \nmid N. \end{cases}$$

where u is the inverse of ℓ modulo N . □

PROOF. Exercise. □

This is hugely important, because it's the underlying input for all of the Euler systems we will build out of Siegel units.

CHAPTER 3

Interlude: motivic cohomology and period integrals

Note: This chapter is provided only for motivation, and involves some very deep and advanced concepts; these will not be needed in the following sections, so you may wish to skip this part at a first reading.

We've seen in the last section that:

- Interesting Galois representations often appear in the étale cohomology (over $\overline{\mathbf{Q}}$) of Shimura varieties.
- One can build classes in H^1 of these Galois representations via Hochschild–Serre, using cup-products, pushforwards from subvarieties, and the Kummer map.
- We have a supply of interesting units on modular curves to use as input to the Kummer map.

For example, if we want to build Euler systems for tensor products $V_f \otimes V_g$, we want classes in $H_{\text{ét}}^3(Y_1(N) \times Y_1(N), \mathbf{Z}_p(2))$; and we can get these by choosing curves $Z \subset Y_1(N)^2$, and pushing forward $\kappa_p(u)$ for some $u \in \mathcal{O}(Z)^\times$. A natural thing to try, of course, is to take Z to be the diagonal copy of $Y_1(N)$, and $u = c g_{0,1/N}$ the Siegel unit.

However, why should this construction give *interesting* classes? How are we going to relate them to the special values of L -functions?

3.1. The Rankin–Selberg integral formula

Here's a very classical result, discovered independently by Rankin and by Selberg in the 1930s.

THEOREM 8. *Let $N \geq 1$, and for $s \in \mathbf{C}$ with $\Re(s) \gg 0$, let E_s be the (non-holomorphic) function on the upper half-plane \mathcal{H} defined by*

$$E_s(\tau) = \pi^{-s} \Gamma(s) \sum_{(c,d) \in \mathbf{Z}^2} \frac{\Im(\tau)^s}{|c\tau + d + 1/N|^{2s}}.$$

Then, for any two newforms f, g of level N and weight 2, we have

$$\langle \bar{f}, gE_s \rangle = \int_{\Gamma_1(N) \backslash \mathcal{H}} f(-\bar{\tau}) g(\tau) E_s(\tau) d\tau \wedge d\bar{\tau} = (*) \cdot L(V_f \otimes V_g, s+1),$$

where $()$ is an explicit factor.*

This is surprisingly simple to prove: after interchanging summation and integration, you get the integral of $f(-\bar{\tau})g(\tau)\Im(\tau)^{-s}$ over the region $\{x+iy : 0 \leq x \leq 1, 0 \leq y \leq \infty\}$, and substituting in the q -expansions of f and g and integrating term-by-term gives the result. However, it has a lot of important consequences; for instance, it follows easily from this formula and the properties of E_s that $L(V_f \otimes V_g, s)$ has meromorphic continuation to all $s \in \mathbf{C}$ (with well-understood poles) and satisfies a functional equation relating s and $3-s$.

However, the reason I want to consider it here is the following classical result (“Kronecker’s second limit formula”)¹

THEOREM 9 (Kronecker). *We have $E_0(\tau) = -\log |g_{0,1/N}|$.*

So there’s some connection between $E_0(\tau)$ and Siegel units, and on the other hand between $E_s(\tau)$ and Rankin–Selberg convolutions. In order to state this properly, we need to introduce another cohomology theory.

3.2. Motivic cohomology

References: Mazza–Voevodsky–Weibel, *Lecture notes on motivic cohomology* [MVW06]; Beilinson, *Higher regulators and values of L -functions* [Beï84].

There is a cohomology theory for algebraic varieties called *motivic cohomology*, introduced by Beilinson and greatly refined by the late Vladimir Voevodsky. It gives groups $H_{\text{mot}}^i(X, \mathbf{Z}(n))$, and for each prime p , there are maps (étale regulators)

$$r_{\text{ét}} : H_{\text{mot}}^i(X, \mathbf{Z}(n)) \otimes \mathbf{Z}_p \rightarrow H_{\text{ét}}^i(X, \mathbf{Z}_p(n)).$$

For small i and n the motivic cohomology groups have explicit descriptions. $H_{\text{mot}}^1(X, \mathbf{Z}(1))$ is literally equal to $\mathcal{O}(X)^\times$, and the étale regulator on this group is the Kummer map κ_p .

REMARK. The étale regulator is compatible with pushforward and cup-products, so in fact our entire toolkit for building elements of étale cohomology factors through motivic cohomology. This also explains why our tools can’t get at $H_{\text{ét}}^i(X, \mathbf{Z}_p(n))$ when $i > 2n$: in this range the group $H_{\text{mot}}^i(X, \mathbf{Z}(n))$ is zero. \diamond

THEOREM 10 (Landsburg [Lan91]). *If S is an algebraic surface over a field k , $H_{\text{mot}}^3(S, \mathbf{Z}(2))$ is isomorphic to the quotient*

$$\left\{ \begin{array}{l} \text{formal sums } \sum_i (Z_i, u_i), \text{ } Z_i \subset S \text{ irreducible curve,} \\ u_i \in k(Z_i)^\times, \text{ with } \sum_i \text{div } u_i = 0 \end{array} \right\} / \sim$$

where \sim is some equivalence relation.

In particular, if we have a curve $Z \subset S$ and an element $u \in \mathcal{O}(Z)^\times$, then $\text{div } u$ is trivial, so (Z, u) defines a class in $H_{\text{mot}}^3(S, \mathbf{Z}(2))$; and (unsurprisingly) the image of this class in $H_{\text{ét}}^3(X, \mathbf{Z}_p(2))$ is just $\iota_* (\kappa_p(u))$, where $\iota : Z \hookrightarrow S$ is the inclusion morphism.

¹I did say this was 19th-century stuff; Kronecker died in 1891.

However, as well as the étale regulator $r_{\text{ét}}$, there’s a second regulator map defined on $H_{\text{mot}}^3(S, \mathbf{Z}(2)) \otimes \mathbf{R}$, the *Beilinson regulator* $r_{\mathbf{C}}$: if ω is a (sufficiently nice) differential 2-form on $S(\mathbf{C})$, we can map an element $\mathfrak{z} = \sum_i (Z_i, u_i)$ to

$$(\dagger) \quad \sum_i \int_{Z_i} \omega \log |u_i|.$$

This is clearly linear in ω , so we get a map from $H_{\text{mot}}^3(S, \mathbf{Z}(2))$ to the dual space of a space of differential forms – more precisely, to $(\text{Fil}^1 H_{\text{dR}}^2(S_{\mathbf{C}}))^*$.

Combining this with what we know about logs of Siegel units, something magical happens: if $S = Y_1(N) \times Y_1(N)$ and \mathfrak{z} is the class of (diagonal, ${}_c g_{0,1/N}$), and we take $\omega = (f(-\bar{\tau}) d\bar{\tau}) \wedge (g(\tau) d\tau)$, then *the integral* (\dagger) *is exactly the Rankin–Selberg integral at $s = 1$!* So, to sum up,

- the class we’ve built in $H_{\text{ét}}^3(Y_1(N)^2, \mathbf{Z}_p(2))$ is naturally the image of something in $H_{\text{mot}}^3(Y_1(N)^2, \mathbf{Z}(2))$,
- the Beilinson regulator of this class, paired with a differential coming from f and g , computes a value of the L -function $L(f \otimes g, s)$.

This is pretty strong evidence that the Galois cohomology class we’re building (the *Beilinson–Flach class*) is the right class to consider: it’s the image under the étale regulator of a motivic class which is a “motivic incarnation” of the Rankin–Selberg integral.

REMARK. It follows from the Beilinson regulator formula that the motivic class $\mathfrak{z} = (\text{diagonal}, {}_c g_{0,1/N}) \in H_{\text{mot}}^3(S, \mathbf{Z}(2))$ is non-zero. If the étale regulator from here to $H_{\text{ét}}^3(S, \mathbf{Z}_p(2))$ were injective, then we could actually deduce that our class in $H_{\text{ét}}^3(S, \mathbf{Z}_p(2))$ was non-zero, and we’d be in a good position to apply Rubin’s theorem.

Sadly, we don’t know this. We can replace S with an integral model \mathcal{S} defined over $\mathbf{Z}[1/pN]$. It’s known that $H_{\text{mot}}^3(\mathcal{S}, \mathbf{Z}(2))/p^k$ maps injectively to $H_{\text{ét}}^3(\mathcal{S}, \mathbf{Z}_p(2))/p^k$ for every k ; but unfortunately we don’t know any finite generation properties for $H_{\text{mot}}^3(\mathcal{S}, \mathbf{Z}(2))$, so \mathfrak{z} might potentially be infinitely p -divisible, and hence zero in $H_{\text{mot}}^3(\mathcal{S}, \mathbf{Z}(2))/p^k$ for every k . It’s conjectured that the motivic cohomology groups of a scheme of finite type over \mathbf{Z} should always be finitely generated, which would rule out this pathology, but unfortunately this conjecture is wide open. \diamond

3.3. Other Rankin–Selberg formulae

The Rankin–Selberg integral is only the first of a very wide class of formulae, which express the L -values of an automorphic form for some reductive group G in terms of its integral against an Eisenstein series on some subgroup H (a “period integral”). There is a survey article by Bump [Bum05] which catalogues dozens of constructions of this kind.

So we can play the following game: if we want to build an Euler system for some class of automorphic Galois representations, then we can look for known formulae expressing the L -function of our representation in terms of periods of automorphic forms. Then we can stare at the resulting integrals and try to recognise them as Beilinson regulators of motivic cohomology classes. If we can do this, then the étale

versions of these classes should be non-zero (although we can't prove this), and they are clearly the right building blocks for an Euler system for our representation.

REMARK. This won't always work, sadly. Firstly, in many of the known Rankin–Selberg formulae the groups G and H do not have Shimura varieties, so they lie outside the world of algebraic geometry; there is a perfectly good Rankin–Selberg integral for $\mathrm{GL}_m \times \mathrm{GL}_n$ for any integers (m, n) , but it doesn't correspond to anything motivic unless $m = n = 2$.

Even if G corresponds to a Shimura variety (and H to a Shimura subvariety), then there can be more subtle obstacles. One major stumbling block is the Eisenstein series appearing in the formulae; these are often not just Eisenstein series for GL_2 but for more general reductive groups, and we need a way to relate these to motivic cohomology, generalising the way that GL_2 Eisenstein series are related to units via Kronecker's limit formula. This seems to be a difficult problem in general.

Despite these apparently gloomy remarks, all is not lost: there are surprisingly many Rankin–Selberg formulae in which only GL_2 Eisenstein series appear! There's now an ongoing project, being pursued by several research groups, to build Euler systems for each such integral formula. Some examples are

- an Euler system for the Asai representation attached to quadratic Hilbert modular forms, with $H = \mathrm{GL}_2$ and $G = \mathrm{Res}_{\mathbf{Q}}^F \mathrm{GL}_2$, where F is a real quadratic field [LLZ16];
- an Euler system for the spin representations attached to genus 2 Siegel modular forms, with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $G = \mathrm{GSp}_4$ ([LSZ17]; we will discuss this example in Chapter 6);
- an Euler system for the spin representation of genus 3 Siegel modular forms, with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ and $G = \mathrm{GSp}_6$, which is studied by Antonio Cauchi and Joaquin Rodrigues [CR18];
- an Euler system for Picard modular forms (work in progress with Chris Skinner), with $H = \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{Res}_{\mathbf{Q}}^K \mathrm{GL}_1$ and $G = \mathrm{GU}(2, 1)$, where K is an imaginary quadratic field and $\mathrm{GU}(2, 1)$ a unitary group split over K . In this case, we get an *Euler system over K* : in other words, we construct cohomology classes over all the finite abelian extensions of K .

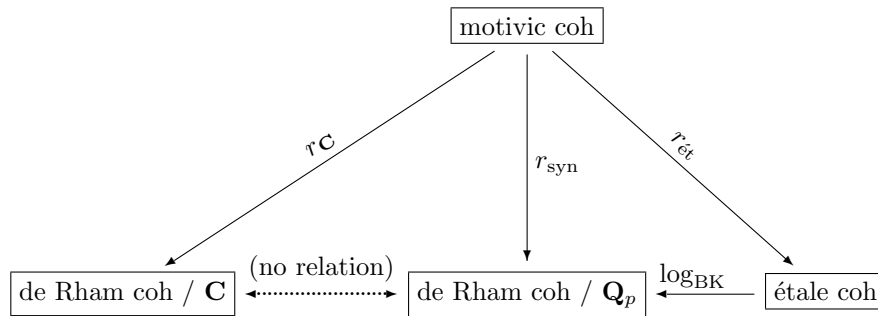
A further example of this is the $\mathrm{GSp}_4 \times \mathrm{GL}_2$ project that we are proposing for this Winter School (see Section 7.1). \diamond

3.4. P-adic regulators

So we have a strategy for building Galois cohomology classes which “really ought to be” non-zero, in the sense that they are the étale images of non-zero motivic cohomology classes. However, since we can't prove that the map from motivic to étale cohomology is injective, how can we be sure these Galois cohomology classes aren't all zero?

To do this, we introduce yet another regulator map defined² on $H^3(S, \mathbf{Z}(2))$, besides the étale and Beilinson regulators: the *p-adic syntomic regulator* [Bes00], which is defined using *p*-adic rigid geometry, assuming $p \nmid N$. The two key properties of this regulator are that

- like Beilinson's, it can be made explicit enough to compute with: there is a formula for the *p*-adic regulator map for a surface, due to Besser [Bes12], which is very closely analogous to (†), with the integral understood via Coleman's *p*-adic integration theory.
- unlike Beilinson's, it can be compared to the étale regulator: a very deep theorem in *p*-adic Hodge theory, due (independently³) to Nizioł and Nekovář [Niz97, Nek98], shows that there is a commutative diagram relating the étale and syntomic regulators via the Bloch–Kato logarithm map of *p*-adic Hodge theory.



Putting these pieces together, if we can build a class $\mathfrak{z} \in H^3(S, \mathbf{Z}(2))$ and show that the syntomic regulator of \mathfrak{z} is non-zero, then its étale regulator must also be non-zero. This programme was carried out in the Rankin–Selberg setting by Bertolini, Darmon and Rotger [BDR15], using Besser's formula [Bes12] to prove that the syntomic regulators of the Beilinson–Flach classes were *p*-adic *L*-values.

²This is not quite true: it is defined on the part of $H^3(S, \mathbf{Z}(2))$ coming from a smooth model S over \mathbf{Z}_p . This is a non-trivial restriction; the work of Flach on adjoint Selmer groups of modular forms relies strongly on the existence of motivic cohomology classes for S which don't extend to S .

³Stronger results have subsequently been proved by these two authors jointly, in [NN16], which treats the case of varieties with bad reduction at p .

The Beilinson–Flach Euler system

In this section we’re going to write down the classes, and prove the “ p -direction” norm relations, for one important example of an Euler system: the Euler system of Beilinson–Flach elements. That is, we’ll define classes over the fields $\mathbf{Q}(\mu_m)$ for all integers m , and we’ll show that if m is of the form p^r , then these classes are compatible under the norm maps for varying r .

References for this lecture: here there is really no alternative to the original papers [LLZ14], [KLZ15] and [KLZ17].

4.1. Beilinson–Flach elements

As we’ve seen in Sections 2.2 and 5.1, we can find this Galois representations attached to Rankin–Selberg convolutions of pairs of weight 2 modular forms in the geometry of $Y_1(N)^2$, for a suitable integer N . Suppose now that both modular forms have weight 2. Then we want to construct classes in the cohomology groups

$$H_{\text{ét}}^3(Y_1(N) \times Y_1(N) \times \mu_m^\circ, \mathbf{Z}_p(2))$$

for $m \geq 1$. Notice that we have only one copy of μ_m° here, not two; so this is best interpreted not as a Shimura variety for $\text{GL}_2 \times \text{GL}_2$, but for the fibre product

$$\text{GL}_2 \times_{\text{GL}_1} \text{GL}_2 = \{(g_1, g_2) \in \text{GL}_2 \times \text{GL}_2 : \det(g_1) = \det(g_2)\}.$$

4.1a. Strategy. In the “Numerology” section above, we saw that one natural line of attack is to find curves $C \subset Y \times Y$, where $Y = Y_1(N)$, and units on C . This approach goes back to Beilinson in 1984 (and was further refined by Flach in 1992, hence the name).

An obvious first guess is to take C to be the diagonally-embedded copy of Y in $Y \times Y$, and then put modular units on C . This is exactly what we’ll do for $m = 1$: we define

$${}_c\text{BF}_{1,N} = \iota_*({}_c g_{0,1/N}),$$

where ι is the diagonal embedding, and $c > 1$ is some integer coprime to everything in sight.

However, how will we get classes over $\mathbf{Q}(\mu_m)$ for $m > 1$? If we had modular units on the curves $Y_1(N) \times \mu_m^\circ$ which were norm-compatible in m , then we could just push these forward in the same way. However, units with this kind of norm-compatibility seem to be hard to find; the Siegel units have very good compatibility properties in the “ N -direction”, but no interesting compatibility in the “ m -direction”.

So we have to make the curve C vary too, and get some contribution to our norm-compatibility this way instead. This is the first hint at a rather powerful general

machine that can turn easy norm relations on a small group into “hard” norm relations on a larger group.

We’ll have a lot of use for the following basic lemma relating pushforward and pullback maps in étale cohomology:

PROPOSITION 5 (Push-pull lemma). *Suppose we have a commutative diagram of morphisms of smooth varieties*

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \beta \downarrow & & \downarrow \gamma \\ Z & \xrightarrow{\delta} & W, \end{array}$$

in which the horizontal maps α and δ are closed embeddings of codimension c , and the vertical maps β and γ are unramified coverings of equal degrees. Then the morphisms $H_{\text{ét}}^i(Z, \mathbf{Z}_p(n)) \rightarrow H_{\text{ét}}^{i+2c}(Y, \mathbf{Z}_p(n+c))$ given by $\alpha_* \circ \beta^*$ and $\gamma^* \circ \delta_*$ coincide. \square

This is a simple instance of a much more general result: the hypotheses imply that the diagram is *Cartesian*, identifying X with the fibre product $Y \times_W Z$. The identity of push-pull and pull-push maps holds for any Cartesian diagram, although we’ll only use diagrams of this simple kind.

4.1b. Mixed-level modular curves.

DEFINITION. *For integers $M \mid N$, let*

$$U(M, N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\widehat{\mathbf{Z}}) : \begin{array}{l} a = 1, b = 0 \pmod{M}, \\ c = 0, d = 1 \pmod{N} \end{array} \right\}.$$

A more compact notation for the same thing, which I’ll use henceforth, is that $U(M, N)$ is the subgroup of level $\begin{pmatrix} M & M \\ N & N \end{pmatrix}$. The definition makes perfect sense without assuming $M \mid N$, of course, but we will only use it in this case. We write $Y(M, N)$ for the corresponding modular curve. Notice that we’ve already seen two special cases: we have $U(1, N) = U_1(N)$, and $U(N, N) = U(N)$.

DEFINITION. *For ℓ prime, we’ll write π_ℓ for the natural quotient map*

$$Y(M, N\ell) \rightarrow Y(M, N).$$

There is also a natural quotient map $Y(M\ell, N) \rightarrow Y(M, N)$, but we won’t use this map. Instead, we’ll be more interested in a “twisted” degeneracy map, which we’ll now define. Let $U(M(\ell), N)$ be the group of level $\begin{pmatrix} M & M\ell \\ N & N \end{pmatrix}$. Then there is a natural quotient map $Y(M\ell, N) \rightarrow Y(M(\ell), N)$; and there are two maps

$$\hat{\pi}_{1,\ell}, \hat{\pi}_{2,\ell} : Y(M(\ell), N) \rightarrow Y(M, N),$$

where $\hat{\pi}_{1,\ell}$ is the natural quotient map, and $\hat{\pi}_{2,\ell}$ corresponds to $\tau \mapsto \tau/\ell$ on \mathcal{H} .

DEFINITION. *We write τ_ℓ for the composite*

$$Y(M\ell, N) \rightarrow Y(M(\ell), N) \xrightarrow{\hat{\pi}_{2,\ell}} Y(M, N).$$

The curve $Y(M, N)$ maps canonically to μ_M° (and in fact the fibres of this map are geometrically connected). We'll abuse notation slightly by writing $Y(M, N)^2$ for the fibre product of two copies of $Y(M, N)$ over their common map to μ_M° . Again, this is most naturally seen as a Shimura variety for $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$.

4.1c. Rankin–Eisenstein classes. The following is an easy check:

PROPOSITION 6. *If $M \mid N$, the group $U(M, N)$ is normalised by the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathbf{Z}})$. \square*

So we can make the following definition:

DEFINITION. *Let $\iota_{M, N}$ be the embedding $Y(M, N) \hookrightarrow Y(M, N)^2$ given by*

$$P \mapsto \left(P, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot P \right).$$

Notice that this corresponds to $\tau \mapsto (\tau, \tau + 1)$ on the upper half-plane.

DEFINITION. *The Rankin–Eisenstein class ${}_c\mathrm{REis}_{M, N}$ is the image of ${}_c g_{0, 1/N}$ under $(\iota_{M, N})_*$.*

4.1d. Beilinson–Flach elements. The final piece of the puzzle is to descend from the higher-level modular curves where the Rankin–Eisenstein classes live to $Y_1(N) \times \mu_M^\circ$. As above, we're identifying $Y_1(N) \times \mu_M^\circ$ with the Shimura variety of level $U' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c = 0, d = 1 \pmod N, ad - bc = 0 \pmod M \right\}$.

One checks easily that

$$\begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} U(M, MN) \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix}^{-1} \subseteq U',$$

so there is a map $s_M : Y(M, MN) \rightarrow Y_1(N) \times \mu_M^\circ$ corresponding to $\tau \mapsto \tau/M$ on \mathcal{H} . This gives us a pushforward map in cohomology, $(s_M \times s_M)_*$.

DEFINITION. *We define the Beilinson–Flach class as the class*

$${}_c\mathrm{BF}_{M, N} = (s_M \times s_M)_* ({}_c\mathrm{REis}_{M, MN}) \in H_{\text{ét}}^3(Y_1(N)^2 \times \mu_M^\circ, \mathbf{Q}_p(2)).$$

These are the classes we really want to study. However, it turns out that proving the norm-compatibility relations for the Beilinson–Flach elements directly is difficult; it's easiest to investigate the norm-compatibility of the auxiliary classes ${}_c\mathrm{REis}_{M, N}$ first, and deduce norm-compatibility relations for the classes ${}_c\mathrm{BF}_{M, N}$ as a consequence. This is what we'll do in the next section.

4.2. Norm-compatibility

It's easy to see that Rankin–Eisenstein classes “inherit” from the Siegel units good norm-compatibility properties in the N -aspect. If ℓ is prime, and π denotes the natural quotient map $Y(M, N\ell) \rightarrow Y(M, N)$ as above, then we have

$$(\pi_\ell \times \pi_\ell)_* ({}_c\mathrm{REis}_{M, \ell N}) = {}_c\mathrm{REis}_{M, N}$$

when $\ell \mid N$ (and there is a slightly modified formula for $\ell \nmid N$).

REMARK. This is a good exercise – you need to use the fact that pushforward maps in étale cohomology are functorial, so that the pushforward map for a composite is the composite of the pushforwards. \diamond

However, they have also, miraculously, acquired an extra norm-compatibility in the M -aspect, which the Siegel units do not have. Recall that $\tau_\ell : Y(M\ell, N) \rightarrow Y(M, N)$ was the “twisted” degeneracy map.

THEOREM 11. *If M, N, ℓ are integers with ℓ prime, $\ell \mid M$ and $M\ell \mid N$, then the Rankin–Eisenstein classes satisfy*

$$(\tau_\ell \times \tau_\ell)_*(c\text{REis}_{\ell M, N}) = (U'(\ell) \times U'(\ell)) \cdot c\text{REis}_{M, N}.$$

Here $U'(\ell)$ is the transpose of the usual Hecke operator $U(\ell)$. The proof of this involves a very important commutative diagram of maps of algebraic varieties over \mathbf{Q} :

$$\begin{array}{ccccc}
 Y(\ell M, N) & \xrightarrow{\iota_{\ell M, N}} & Y(\ell M, N)^2 & & \\
 \parallel & & \downarrow & \searrow & \\
 Y(\ell M, N) & \xrightarrow{\iota'} & Y(M(\ell), N)^2 & & \\
 \downarrow & & \downarrow \hat{\pi}_{1, \ell} \times \hat{\pi}_{1, \ell} & \searrow & \\
 & \diamond & & & \\
 Y(M, N) & \xrightarrow{\iota_{M, N}} & Y(M, N)^2 & & \\
 & & \downarrow \hat{\pi}_{2, \ell} + \hat{\pi}_{2, \ell} & \searrow & \\
 & & & & Y(M, N)^2
 \end{array}$$

Here the two diagonal maps are the ones introduced in the previous section, and the vertical maps are the natural quotient maps. The commutativity of the diagram is obvious by construction; the two really important and nonobvious properties are the following:

PROPOSITION 7. *Under the hypotheses of the theorem, the map ι' is a closed embedding, and the lower left square marked \diamond is a Cartesian diagram of the kind described in Proposition 5.*

PROOF. It’s easy to see that the image of ι' is precisely the modular curve associated to the group

$$U(M(\ell), N) \cap \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} U(M(\ell), N) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

However, a straightforward matrix computation shows that this intersection is nothing but $U(\ell M, N)$ itself. So ι' is a closed embedding.

Since both horizontal maps in the square \diamond are closed embeddings, and the vertical maps are automatically finite coverings, it suffices to check that the degrees of the vertical maps agree. These degrees are equal to the indices of corresponding inclusions of level groups: on the left-hand side we have $[U(M, N) : U(\ell M, N)] = \ell^2$, and on the right-hand side $[U(M, N)^2 : U(M(\ell), N)^2] = \ell^2$. \square

Exercise. Show that if M, N, ℓ are integers with ℓ prime, $\ell \nmid M$ and $M\ell \mid N$, then the Rankin–Eisenstein classes satisfy

$$(5) \quad (\tau_\ell \times \tau_\ell)_* ({}_c\text{REis}_{\ell M, N}) = (U'(\ell) \times U'(\ell) - \Delta_\ell^*) \cdot {}_c\text{REis}_{M, N},$$

where Δ_ℓ denotes any element of $\text{GL}_2(\mathbf{Z}/MN\mathbf{Z})^2$ of the form $\left(\begin{pmatrix} x & \\ & 1 \end{pmatrix}, \begin{pmatrix} x & \\ & 1 \end{pmatrix} \right)$ with $x \equiv \ell \pmod{m}$.

COROLLARY 2. *The following two classes in $H_{\text{ét}}^3(Y(M(\ell), N)^2, \mathbf{Z}_p(2))$ coincide:*

- *the pushforward of ${}_c\text{REis}_{\ell M, N}$ along the upper vertical arrow $Y(\ell M, N)^2 \rightarrow Y(M(\ell), N)^2$;*
- *the pullback of ${}_c\text{REis}_{M, N}$ along the lower vertical arrow $Y(M(\ell), N)^2 \rightarrow Y(M, N)^2$.*

PROOF. This is exactly the “push-pull” lemma applied to the square \diamond (since the unit ${}_c g_{0,1/N}$ on $Y(\ell M, N)$ is, by definition, the pullback of the unit with the same name on $Y(M, N)$.) \square

Since these two classes are equal on $Y(M(\ell), N)^2$, they certainly must have the same pushforward along the diagonal map to $Y(M, N)^2$. So we obtain an equality between $(\tau_\ell \times \tau_\ell)_* ({}_c\text{REis}_{\ell M, N})$ and the image of ${}_c\text{REis}_{\ell M, N}$ under pullback and pushforward around the triangle. This composite of pushforward and pullback maps is exactly the Hecke operator $U'(\ell) \times U'(\ell)$, so we have proved the theorem. \square

REMARK. The proof works with a minor modification if we drop the hypothesis that $\ell \mid M$. If $\ell \nmid M$, then ι' is still a closed embedding, but the degrees of the vertical maps in the square are $\ell(\ell - 1)$ on one side and ℓ^2 on the other. In order to obtain a Cartesian square, we have to modify \diamond by replacing $Y(\ell M, N)$ with the disjoint union $Y(\ell M, N) \sqcup Y(M(\ell), N)$ which does have degree ℓ^2 over $Y(M, N)$. Some simple bookkeeping later we conclude that

$$(\tau_\ell \times \tau_\ell)_* ({}_c\text{REis}_{\ell M, N}) = [(U'(\ell) \times U'(\ell)) - \sigma_\ell] \cdot {}_c\text{REis}_{M, N}.$$

On the other hand, the assumption $\ell M \mid N$ is essential, since otherwise the definition of the Rankin–Eisenstein element doesn’t even make sense. \diamond

We can now state and prove the main theorem:

THEOREM 12. *If ℓ is prime with $\ell \mid M$ and $\ell \mid N$, and $j \in (\mathbf{Z}/\ell M\mathbf{Z})^\times$, we have*

$$\text{norm}_{\mathbf{Q}(\mu_M)}^{\mathbf{Q}(\mu_{\ell M})} ({}_c\text{BF}_{\ell M, N}) = [U'(\ell) \times U'(\ell)] \cdot {}_c\text{BF}_{M, N}.$$

PROOF. This follows from the commutativity of the diagram

$$\begin{array}{ccccc}
 Y(\ell M, \ell MN) & \xrightarrow{\tau_\ell} & Y(M, \ell MN) & \xrightarrow{\pi_\ell} & Y(M, MN) \\
 \downarrow s_{\ell M} & & & & \downarrow s_M \\
 Y_1(N) \times \mu_{\ell M}^\circ & \longrightarrow & & \longrightarrow & Y_1(N) \times \mu_M^\circ,
 \end{array}$$

and the following compatibilities:

- the main theorem of the previous section, which we use to compare $\text{REis}_{\ell M, \ell MN}$ with $\text{REis}_{M, M\ell N}$;
- the norm-compatibility in N , which allows us to compare $\text{REis}_{M, M\ell N}$ with $\text{REis}_{M, MN}$;
- the fact that $U'(\ell)$ commutes with the pushforward along the maps π_ℓ and s_M . \square

Exercise. Using (5), formulate and prove the analogous statement in the case when $\ell \nmid M$ and $\ell \mid N$.

REMARKS.

- (i) It is also possible to describe the class ${}_c\text{BF}_{M,N}$ directly at level N (rather than going via the higher-level curves $Y(M, MN)$ as we have done). The curve image $(\iota_{M, MN}) \subset Y(M, MN)^2$ maps down via $s_M \times s_M$ to a curve $C_{M,N} \subset Y_1(N)^2 \times \mu_M^\circ$, and our class can be characterised as the pushforward of a unit on $C_{M,N}$. However, the curve $C_{M,N}$ is rather messy (it can have many self-intersections, for instance), which makes it more difficult to prove the norm relation by this approach.
- (ii) The compatibility of $U'(\ell)$ with pushforwards may seem like a minor point, but I want to emphasise it here, because this is the point where the proof breaks down in the case $\ell \nmid MN$. In this case, there is an operator $U'(\ell)$ on $Y(M, \ell MN)$, and an operator $T'(\ell)$ on $Y(M, MN)$, but these aren't compatible under π_* . So to complete the argument we would need to relate

$$(\pi \times \pi)_* \left[(U'(\ell) \times U'(\ell)) \cdot {}_c\text{REis}_{M, N\ell} \right]$$

to the objects we know about on $Y(M, MN)$. This can be done – in fact there are at least three separate approaches – but it isn't easy. The eventual outcome is that for $\ell \nmid MN$ we have a formula

$$\text{norm}_{\mathbf{Q}(\mu_M)}^{\mathbf{Q}(\mu_{\ell M})} ({}_c\text{BF}_{\ell M, N}) = Q_\ell(\sigma_\ell^{-1}) \cdot {}_c\text{BF}_{M, N},$$

where $Q_\ell(X)$ is a degree 4 polynomial with coefficients in the Hecke algebra. \diamond

4.3. Projection to the (f, g) component

We now bring the eigenforms f and g into the picture. It's important to impose some local conditions at p . We take f and g to be eigenforms of some level N , with $p \mid N$, whose $U(p)$ -eigenvalues α_f, α_g are p -adic units (we say f and g are *ordinary* at p).

REMARK. If we start with some form f of level N_0 with $p \nmid N_0$, then we replace f with one of the two $U(p)$ -eigenforms of level $N = pN_0$ which have the same Hecke eigenvalues away from p . This process is called **p -stabilisation**. This doesn't change the Galois representations: the Galois representations attached to the p -stabilisations of f are isomorphic to that of the original form f , although they live on a different modular curve. \diamond

The quotient

$$H_{\text{ét}}^1 \left(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(1) \right) / \left\langle \begin{array}{l} T'(\ell) - a_\ell(f) \forall \ell \nmid N, \\ U'(\ell) - a_\ell(f) \forall \ell \mid N \end{array} \right\rangle$$

turns out to be isomorphic to the dual¹ V_f^* of V_f . The image of the cohomology with \mathbf{Z}_p -coefficients gives a lattice T_f^* in V_f^* . Doing this for both f and g , and combining this with the Hochschild–Serre spectral sequence, we get a projection map

$$\text{Pr}_{f,g} : H_{\text{ét}}^3(Y_1(N) \times \mu_M^\circ, \mathbf{Z}_p(2)) \rightarrow H^1(\mathbf{Q}(\mu_M), T_f^* \otimes T_g^*).$$

By construction, the Hecke operator $U'(\ell) \times U'(\ell)$ on the source corresponds to multiplication by $\alpha_f \alpha_g$ on the target. This gives us the following theorem:

PROPOSITION 8. *The classes*

$$(\alpha_f \alpha_g)^{-r} \text{Pr}_{f,g}(c\text{BF}_{p^r, N, 1}) \in H^1(\mathbf{Q}(\mu_{p^r}), T_f^* \otimes T_g^*)$$

are norm-compatible for $r \geq 1$. \square

Notice that it's crucial that α_f, α_g are p -adic units, since otherwise these renormalised classes wouldn't land in $T_f^* \otimes T_g^*$ any more.

Exercise. Show that if $p \mid N$, then

$$\text{cores}_{\mathbf{Q}}^{\mathbf{Q}(\mu_p)} \text{Pr}_{f,g}(c\text{BF}_{p, N}) = (\alpha_f \alpha_g - 1) \text{Pr}_{f,g}(c\text{BF}_{1, N}).$$

The exercise shows that the case $r = 0$ doesn't quite work; there is an unwanted Euler factor appearing, just as in the case of cyclotomic units. Exactly as in that case, we can get rid of this error term by re-defining the $r = 0$ class to be the norm of the $r = 1$ class. This gives an element of the module

$$H_{\text{Iw}}^1(\mathbf{Q}(\mu_{p^\infty}), T_f^* \otimes T_g^*) := \varprojlim_{r \geq 0} H^1(\mathbf{Q}(\mu_{p^r}), T_f^* \otimes T_g^*),$$

which is the *Iwasawa cohomology* of $T_f^* \otimes T_g^*$.

¹The dual appears here because the $T'(\ell)$ are the adjoints of the $T(\ell)$ under Poincaré duality.

Modular forms of higher weight

5.1. Galois representations

When we defined Galois representations attached to modular forms, we assumed that the modular forms has weight 2. Let's now see how this extends to other weights.

Assume now that f is a cuspidal modular eigenform of level $\Gamma_1(N)$ and weight $k+2$ for some $k \geq 0$. (We say that f has *cohomological weight*.) It turns out that we can still attach a Galois representation to f , but if $k > 0$, then we have to consider *étale cohomology with coefficients*.

It follows from Theorem 6 that there is a universal elliptic curve over $Y_1(N)$, say $\pi : \mathcal{E} \rightarrow Y_1(N)$. Denote by \mathcal{H} the étale sheaf $V_p(\mathcal{E})$ on $Y_1(N)$; this is a locally constant sheaf of \mathbf{Q}_p -vector spaces of dimension 2, whose fibre at any geometric point x is canonically identified with the Tate module $V_p(\mathcal{E}_x)$ of the elliptic curve $\mathcal{E}_x/\overline{K}$.

REMARK. We have a functor

$$(\star) \quad \{\text{algebraic representations of } \mathrm{GL}_2/\mathbf{Q}_p\} \rightarrow \{\text{étale } \mathbf{Q}_p\text{-sheaves on } Y_1(N)\}.$$

The sheaf \mathcal{H} is the image under this functor of the defining 2-dimensional representation of GL_2 . \diamond

DEFINITION. We let $V_p(f)$ be the largest subspace of $H_{\text{ét}}^1(Y_1(N)_{\overline{\mathbf{Q}}}, \mathrm{Sym}^k \mathcal{H}(-k))$ on which the Hecke operators $T(\ell)$, for $\ell \nmid N$, act as multiplication by $a_\ell(f)$.

Then $V_p(f)$ has the expected properties (generalising those we had above for $k = 0$):

- (1) $V_p(f)$ is 2-dimensional and irreducible.
- (2) $V_p(f)$ is a direct summand of $H_{\text{ét}}^1$ (not just a subspace).
- (3) For $\ell \nmid pN$, $V_p(f)$ is unramified at ℓ and the local Euler factor is

$$P_\ell(V_p(f), t) = 1 - a_\ell(f)t + \ell^{k+1}\chi(\ell)t^2.$$

- (4) $V_p(f)^* = V_p(f \otimes \chi^{-1})(k+1)$.

REMARK. There are also Galois representations attached to weight 1 modular forms, but these are harder to construct – they don't show up in étale cohomology with coefficients in any reasonable sheaf. \diamond

In much the same way, if we have a pair of integers $(k, k') \geq 0$ we can form a sheaf $\mathrm{Sym}^k \mathcal{H} \boxtimes \mathrm{Sym}^{k'} \mathcal{H}$ on $Y_1(N)^2$, and the tensor product $V(f) \otimes V(g)$, for f and g eigenforms of weight $k+2$ and $k'+2$, appears as a direct summand of the space

$$H_{\text{ét}}^2\left(Y_1(N)_{\overline{\mathbf{Q}}}^2, (\mathrm{Sym}^k \mathcal{H} \boxtimes \mathrm{Sym}^{k'} \mathcal{H})(-k-k')\right).$$

5.2. Eisenstein classes

The Kummer images of Siegel units give us classes in $H_{\text{ét}}^1(Y_1(N), \mathbf{Q}_p(1))$. What should our higher-weight analogues of this be?

It turns out that for any $k \geq 0$ and $N \geq 4$, and $c > 1$ coprime to $6pN$, there exists an *étale Eisenstein class*

$${}_c \text{Eis}_{0,1/N}^k \in H_{\text{ét}}^1\left(Y_1(N), \text{Sym}^k \mathcal{H}(1)\right),$$

which in the case $k = 0$ agrees with the Kummer-map image of the Siegel unit. These étale Eisenstein symbols satisfy similar basic relations (Theorem 7) to those of the Siegel units.

REMARK. One can make sense of “motivic cohomology with coefficients in \mathcal{H} ”, and then one finds that these Eisenstein classes are the étale images of motivic Eisenstein classes, whose images under the Beilinson regulator are non-holomorphic Eisenstein series of weight $-k$. This is a higher-weight generalisation of the Kronecker limit formula, since for $k = 0$ the Beilinson regulator map on $H_{\text{mot}}^1(Y_1(N), \mathbf{Z}(1)) \cong \mathcal{O}(Y_1(N))^\times$ maps a unit u to the function $\log |u| : Y_1(N)(\mathbf{C}) \rightarrow \mathbf{R}$. \diamond

5.3. The Euler system for higher weight modular forms

We can adapt the above construction for pairs of modular forms of higher weight. Suppose that f and g have weights $k + 2$ and $k' + 2$ with $k, k' \geq 2$. Then it follows from Section 5.1, we need to construct classes in the cohomology groups

$$H_{\text{ét}}^3\left(Y_1(N) \times Y_1(N) \times \mu_m^\circ, \text{Sym}^k \mathcal{H} \boxtimes \text{Sym}^{k'} \mathcal{H}(n)\right),$$

for some appropriate $n \in \mathbf{Z}$, and these classes should arise via pushforward from the cohomology of $Y_1(N)$. Assume that $m = 1$, so we want to pushforward along the diagonal embedding $\iota : Y_1(N) \rightarrow Y_1(N)^2$. (Once we have understood this case, we can construct classes for $m > 1$ using the methods from the previous sections.)

5.3a. Pushforward with coefficients. It turns out that pushforward maps “work” with coefficients: there’s a natural map

$$\iota_* : H_{\text{ét}}^1(Y_1(N), \iota^*(\mathcal{L})(1)) \rightarrow H_{\text{ét}}^3(Y_1(N)^2, \mathcal{L}(2))$$

for any étale sheaf \mathcal{L} , with the case above being \mathcal{L} the constant sheaf \mathbf{Q}_p . Here $\iota^* \mathcal{L}$ is just the pullback of \mathcal{L} to $Y_1(N)$. So what does the sheaf $\iota^*\left(\text{Sym}^k \mathcal{H} \boxtimes \text{Sym}^{k'} \mathcal{H}\right)$ look like?

Since \mathcal{H} and its symmetric powers arise from irreducible algebraic representations of GL_2 , we can use group theory to answer this question. Let V denote the standard 2-dimensional \mathbf{Q}_p -representation of GL_2 . Then the sheaf $\text{Sym}^k \mathcal{H} \boxtimes \text{Sym}^{k'} \mathcal{H}$ on $Y_1(N)^2$ arises from the irreducible representation $\text{Sym}^k V \boxtimes \text{Sym}^{k'} V$ of $G := \text{GL}_2 \times_{\text{GL}_1} \text{GL}_2$. If $H \subset G$ denotes the diagonally-embedded copy of GL_2 , then the restriction of this G -representation to H breaks up as a sum of irreducible H -representations; and we have a corresponding decomposition of the pullback $\iota^*\left(\text{Sym}^k \mathcal{H} \boxtimes \text{Sym}^{k'} \mathcal{H}\right)$ in the category of sheaves on $Y_1(N)$.

REMARK. A posh way of stating this compatibility is that we have a commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Rep}_G & \longrightarrow & \text{sheaves}/Y_1(N)^2 \\ \mathrm{res}_H^G \downarrow & & \downarrow \iota^* \\ \mathrm{Rep}_H & \longrightarrow & \text{sheaves}/Y_1(N) \end{array}$$

where Rep_G and Rep_H are the categories of representations of G and its subgroup H , res_H^G is restriction of representations, and the horizontal arrows are the functors (\star) for G and H . An analogue of this naturality property has been established for motivic cohomology in recent works of Ancona and Torzewski. \diamond

The decomposition of $\mathrm{res}_H^G \left(\mathrm{Sym}^k V \boxtimes \mathrm{Sym}^{k'} V \right) = \mathrm{Sym}^k V \otimes \mathrm{Sym}^{k'} V$ into irreducible representations of H is described by the *Clebsch–Gordan formula*:

$$\mathrm{Sym}^k V \otimes \mathrm{Sym}^{k'} V = \bigoplus_{j=0}^{\min\{k,k'\}} \mathrm{Sym}^{k+k'-2j} V \otimes \det^j,$$

so for every $0 \leq j \leq \min\{k, k'\}$, we have a GL_2 -equivariant map

$$\mathrm{Sym}^{k+k'-2j} V \otimes \det^j \longrightarrow \iota^* \left(\mathrm{Sym}^k V \boxtimes \mathrm{Sym}^{k'} V \right).$$

The representation \det^j of GL_2 corresponds to the sheaf $\mathbf{Q}_p(j)$, so this means that for every $0 \leq j \leq \min\{k, k'\}$ we get a map of sheaves on $Y_1(N)$,

$$\mathrm{Sym}^{k+k'-2j} \mathcal{H} \longrightarrow \iota^* \left(\mathrm{Sym}^k \mathcal{H} \boxtimes \mathrm{Sym}^{k'} \mathcal{H}(-j) \right),$$

which induces a map in étale cohomology

$$\iota_* : H_{\text{ét}}^1 \left(Y_1(N), \mathrm{Sym}^{k+k'-2j} \mathcal{H}(1) \right) \longrightarrow H_{\text{ét}}^3 \left(Y_1(N)^2, \mathrm{Sym}^k \mathcal{H} \boxtimes \mathrm{Sym}^{k'} \mathcal{H}(2-j) \right).$$

5.3b. Definition of the classes. As we saw above, there is a special element in $H_{\text{ét}}^1 \left(Y_1(N), \mathrm{Sym}^{k+k'-2j} \mathcal{H}(1) \right)$, the étale Eisenstein class ${}_c \mathrm{Eis}_{0,1/N}^{(k+k'-2j)} \in H_{\text{ét}}^1 \left(Y_1(N), \mathrm{Sym}^{k+k'-2j} \mathcal{H}(1) \right)$.

DEFINITION. Let $0 \leq j \leq \min\{k, k'\}$. We define the *Rankin–Eisenstein class*

$${}_c \mathrm{REis}_{1,N}^{(k,k',j)} = \iota_* \left({}_c \mathrm{Eis}_{0,1/N}^{(k+k'-2j)} \right),$$

which is an element of $H_{\text{ét}}^3 \left(Y_1(N)^2, \mathrm{Sym}^k \mathcal{H} \boxtimes \mathrm{Sym}^{k'} \mathcal{H}(2-j) \right)$.

Using the same methods as in Sections 4.1c and 4.1d, we more generally define Rankin–Eisenstein classes ${}_c \mathrm{REis}_{M,N}^{(k,k',j)}$ for $M|N$; and (finally) Beilinson–Flach classes

$${}_c \mathrm{BF}_{m,N}^{(k,k',j)} \in H_{\text{ét}}^3 \left(Y_1(N)^2 \times \mu_m^\circ, \mathrm{Sym}^k \mathcal{H} \otimes \mathrm{Sym}^{k'} \mathcal{H}(2-j) \right)$$

as the image of ${}_c \mathrm{REis}_{m,mN}^{(k,k',j)}$ under the map¹ $(s_m \times s_m)_*$.

¹One needs to be a bit careful here with extending s_m to a map on cohomology with coefficients, but we don't discuss this issue here. For reference, see [KLZ17, §6.1].

Now let f and g be eigenforms of weights $k + 2, k' + 2 \geq 2$ with $k, k' \geq 0$ and level N , where $p \mid N$, both of which are ordinary at p . It then follows from Section 5.1 and the arguments in Section 4.3 that we have a projection map

$$\begin{aligned} \Pr_{f,g} : H_{\text{ét}}^3 \left(Y_1(N)^2 \times \mu_m^\circ, \text{Sym}^k \mathcal{H} \otimes \text{Sym}^{k'} \mathcal{H}(2-j) \right) \\ \longrightarrow H^1(\mathbf{Q}(\mu_m), (V_f \otimes V_g)^*(-j)), \end{aligned}$$

and as in the parallel weight 2 case one can show that the images of the Beilinson–Flach classes $\left({}_c \text{BF}_{m,N}^{(k,k',j)} \right)_{m \geq 1}$ under this projection map have the same properties under corestriction maps as in Proposition 8.

REMARK. We also have to check that these objects land in a \mathbf{Z}_p -lattice independent of m . To do this, we need to find good integral lattices in $\text{Sym}^k \mathcal{H}$. There is a natural \mathbf{Z}_p -lattice subsheaf $\mathcal{H}_{\mathbf{Z}_p} \subset \mathcal{H}$, but a small complication arises because with \mathbf{Z}_p -coefficients the Sym^k functor is not compatible with duality unless $p > k$. To repair this one has to introduce a slightly different sheaf, the sheaf $\text{TSym}^k \mathcal{H}_{\mathbf{Z}_p}$ of symmetric tensors, which is only isomorphic to $\text{Sym}^k \mathcal{H}_{\mathbf{Z}_p}$ if $p > k$. \diamond

5.4. Twist-compatibility

The upshot of this construction is that for a fixed pair of forms f and g , we have not one but $1 + \min\{k, k'\}$ *different* Euler systems, which live in different cyclotomic twists of the representation $V_f^* \otimes V_g^*$. However, as we've seen above, Soulé's twisting construction gives an isomorphism between the space of Euler systems for V and for $V(m)$ for any $m \in \mathbf{Z}$, so it makes sense to compare these Euler systems to each other.

THEOREM 13. *The Beilinson–Flach Euler systems associated to different values of j in the range $0 \leq j \leq \min\{k, k'\}$ are all compatible under the Soulé twist.*

This simple-looking statement turns out to be deceptively hard. See [KLZ17, §6].

REMARK. A similar issue arises for Kato's Euler system associated to a single modular form, and in this case one even has *infinitely many* potentially different Euler systems! More precisely, for a weight 2 form f , one can use cup-products of two weight n Eisenstein classes, for any $n \geq 0$, to construct an Euler system with values in $V_p(f)^*(1+n)$.

Naturally, one expects that the Euler system thus constructed for any $n \geq 1$ should coincide with the n -th Soulé twist of the $n = 0$ Euler system. This was checked in the PhD thesis of Matthew Gealy [Gea06]. \diamond

5.5. An adelic modification

Just in order to motivate some of the constructions we'll use in later chapters, it's worth pointing out that one can make a slight modification to the construction. Since we have defined our modular curves $Y(U)$ as quotients of $\text{GL}_2(\mathbf{A}_f) \times \mathcal{H}$ (c.f. (1)), where \mathbf{A}_f are the finite adèles, we have a (right) action of the normaliser of $\text{GL}_2(\mathbf{A}_f)$ on the tower of curves $Y(U)$ for varying U . This is compatible with the

action of $\mathrm{GL}_2^+(\mathbf{Q}) \subset \mathrm{GL}_2(\mathbf{A}_f)$ via Möbius transformations on \mathcal{H} , after modifying by an inverse to interchange left and right actions.

With these conventions, we can define our Hecke operators, and our degeneracy maps τ_ℓ , s_M etc, using elements of $\mathrm{GL}_2(\mathbf{A}_f)$ which are the identity outside the place ℓ . This does not change anything major (the difference between the “old” and “new” elements is given by the action of an element of $\mathrm{Gal}(\mathbf{Q}(\mu_m)/\mathbf{Q})$) but the adelic presentation makes it a little easier to leverage results from representation theory.

CHAPTER 6

An Euler system for Siegel modular forms

We will describe the construction of the Euler system using the adelic approach, as described in Section 5.5. This is consistent with the approach taken in the main reference for this chapter [LSZ17].

6.1. Siegel modular 3-folds

DEFINITION. Let J be the skew-symmetric 4×4 -matrix $\begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$. Define GSp_4 to be the group scheme over \mathbf{Z} such that for any \mathbf{Z} -algebra R , we have

$$\mathrm{GSp}_4(R) = \{(g, \nu) \in \mathrm{GL}_4(R) \times R^\times : gJg^t = \nu J\}.$$

We let Sp_4 be the subgroup of elements with $\nu = 1$.

The group $\mathrm{GSp}_4^+(\mathbf{R})$ (the elements of $\mathrm{GSp}_4(\mathbf{R})$ with $\nu > 0$) acts on the genus 2 Siegel upper half space

$$\mathcal{H}_2 = \{Z \in M_2(\mathbf{C}) : Z = \begin{pmatrix} y & z \\ x & y \end{pmatrix}, \quad \Im\left(\begin{pmatrix} x & y \\ y & z \end{pmatrix}\right) \text{ is positive definite}\}$$

via $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$.

REMARK. If we use a slightly different model of GSp_4 , as matrices satisfying $gJ'g^t = \nu J'$ where $J' = \begin{pmatrix} & & & 1 \\ & & I_2 & \\ & -I_2 & & \\ & & & \end{pmatrix}$, then we can define \mathcal{H}_2 more tidily, as the space of symmetric complex matrices with positive-definite imaginary part. However, defining GSp_4 using the anti-diagonal matrix J , as we have done, is more convenient for representation theory (as the intersection of GSp_4 with the upper-triangular matrices in GL_4 is a Borel subgroup). \diamond

If U is an open compact subgroup of $\mathrm{GSp}_4(\mathbf{A}_f)$, then we can define the double quotient

$$\tilde{Y}(U) = \mathrm{GSp}_4^+(\mathbf{Q}) \backslash (G(\mathbf{A}_f) \times \mathcal{H}_2) / U.$$

This is a 3-dimensional complex manifold, with finitely many components, each of which looks like $\Gamma \backslash \mathcal{H}_2$ for some discrete subgroup $\Gamma \subset \mathrm{Sp}_4(\mathbf{Q})$.

THEOREM 14. *If U is sufficiently small, then $\tilde{Y}(U)$ is the \mathbf{C} -points of a smooth algebraic variety $\tilde{Y}(N)$ defined over \mathbf{Q} (a Siegel 3-fold), which is a moduli space for principally polarised abelian surfaces with some level structure.*

Of course, the kind of level structure that emerges depends on the group U we choose. A particularly important case is when

$$U = U_1(N) := \{(g, \nu) \in \mathrm{GSp}_4(\hat{\mathbf{Z}}) : g = \begin{pmatrix} * & & & \\ & * & & \\ & & I_2 & \\ & & & * \end{pmatrix} \bmod N\}$$

(where 0_2 and I_2 are the 2×2 zero and identity matrices respectively). The corresponding threefold $\tilde{Y}_1(N)$ parametrises triples (A, λ, P, Q) where A is an abelian surface, λ is a principal polarisation on A , and $P, Q \in A[N]$ are two points of exact order N satisfying $\langle P, Q \rangle = 0$ (where \langle, \rangle is the Weil pairing induced by the polarisation λ).

As in the case of modular curves, we can identify the basechange $\tilde{Y}_1(N) \times \mu_m^\circ$ with a Shimura variety $\tilde{Y}(U)$ for some modified level m . More precisely, if we let

$$\mathcal{U} = \{(g, \nu) \in U_1(N) : \nu = 1 \pmod{m}\}$$

then $\tilde{Y}(\mathcal{U})$ is canonically isomorphic to $\tilde{Y}_1(N) \times \mu_m^\circ$ as a \mathbf{Q} -variety.

REMARK. In terms of moduli spaces, the projection to μ_m° is given by the Weil pairing. \diamond

6.2. Genus 2 Siegel modular forms

References: van der Geer's article [vdG08] is an excellent introduction; more details (particularly on Hecke operators) can be found in Andrianov's book [And87].

6.2a. Definitions.

DEFINITION. Let $\tilde{\Gamma}_1(N) = \mathrm{Sp}_4(\mathbf{Z}) \cap \tilde{U}_1(N)$. A Siegel modular form of genus 2, level N and weight (k, k) is a holomorphic function $\mathcal{F} : \mathcal{H}_2 \rightarrow \mathbf{C}$ such that $\mathcal{F}(g \cdot Z) = \det(CZ + D)^k \mathcal{F}(Z)$ for all $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}_1(N)$ and $Z \in \mathcal{H}_2$. We write $M_{k,k}^{(2)}(\tilde{\Gamma}_1(N))$ for the space of such functions.

Note the similarity to the familiar definition of modular forms (which are automorphic forms for $\mathrm{GSp}_2 \cong \mathrm{GL}_2$).

REMARK. There is a more general notion of Siegel modular forms of weight (k_1, k_2) for integers $k_1 \geq k_2$; these are holomorphic functions on \mathcal{H}_2 , taking values the space $\mathbf{C}^{k_1 - k_2 + 1}$, and the transformation law involves the action of $CZ + D$ via the representation $\mathrm{Sym}^{k_1 - k_2} \otimes \det^{k_2}$ of $\mathrm{GL}_2(\mathbf{C})$. When $k_1 > k_2$ these are sometimes called *vector-valued Siegel modular forms*, and the forms for $k_1 = k_2$ are called *scalar-valued*. \diamond

As for usual modular forms, the space $M_{k_1, k_2}^{(2)}(\tilde{\Gamma}_1(N))$ is finite-dimensional over \mathbf{C} , and has a subspace $S_{k_1, k_2}^{(2)}(\tilde{\Gamma}_1(N))$ of *cuspidal* forms.

6.2b. Hecke operators. We can also describe $M_{k_1, k_2}^{(2)}(\tilde{\Gamma}_1(N))$ and $S_{k_1, k_2}^{(2)}(\tilde{\Gamma}_1(N))$ adelically, using the isomorphism

$$\tilde{\Gamma}_1(N) \backslash \mathcal{H}_2 \cong \mathrm{GSp}_4^+(\mathbf{Q}) \backslash (\mathrm{GSp}_4(\mathbf{A}_f) \times \mathcal{H}_2) / U_1(N).$$

From this interpretation, we get an action on these spaces of the *Hecke algebra* of double cosets $U_1(N)gU_1(N)$, $g \in \mathrm{GSp}_4(\mathbf{A}_f)$. This decomposes as a product of local Hecke algebras for each prime ℓ .

For $\ell \nmid N$, the local Hecke algebra is generated by three operators $T(\ell)$, $T_1(\ell^2)$, and $R(\ell)$, corresponding to the double cosets of $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ell & \\ & & & \ell \end{pmatrix}$, $\begin{pmatrix} 1 & & & \\ & \ell & & \\ & & \ell & \\ & & & \ell^2 \end{pmatrix}$, and $\begin{pmatrix} \ell & & & \\ & \ell & & \\ & & \ell & \\ & & & \ell \end{pmatrix}$

(considered as elements of $\mathrm{GSp}_4(\mathbf{Q}_\ell) \subset \mathrm{GSp}_4(\mathbf{A}_f)$, with components at all places other than ℓ being the identity).

DEFINITION. If $\mathcal{F} \in S_{k_1, k_2}^{(2)}(\tilde{\Gamma}_1(N))$ is an eigenform for the above three operators, with eigenvalues $t(\ell), t_1(\ell^2), r(\ell)$ respectively, then the spin L -factor of \mathcal{F} at ℓ is the degree 4 polynomial

$$P_{\mathrm{spin}, \ell}(\mathcal{F}, X) = 1 - t(\ell)X + \ell(t_1(\ell^2) + (\ell^2 + 1)r(\ell))X^2 - \ell^3 t(\ell)r(\ell)X^3 + \ell^6 r(\ell)^2 X^4.$$

(Often we work with the renormalised polynomial $P_{\mathrm{spin}, \ell}(\mathcal{F}, \ell^{-3/2}X)$, which has the advantage that its roots all have absolute value 1.)

This is, of course, crying out to be made into an Euler product

$$L_{\mathrm{spin}}(\mathcal{F}, s) = \prod_{\ell \text{ prime}} P_{\mathrm{spin}, \ell} \left(\mathcal{F}, \ell^{-s - \frac{3}{2}} \right)^{-1},$$

the *spin L -function* of \mathcal{F} , although this only makes sense if we have a good definition of the local factors at primes $\ell \mid N$. Under some mild hypotheses on \mathcal{F} , a suitable recipe for these factors was found by Piatetski-Shapiro and Novodvorsky in the 1970s (although not published until 1997 [PS97]), and they showed that the resulting function has meromorphic continuation with a functional equation of the form $s \mapsto 1 - s$.

REMARK. As well as the spin L -function, there is another L -function associated to \mathcal{F} , confusingly called the *standard L -function*, given by a different Euler product in which the local factors at the good primes are reciprocals of polynomials in ℓ^{-s} of degree 5. The terminology “standard” is unfortunate for GSp_4 , since the spin L -function is a much more fundamental object than the standard one, but it reflects the fact that the standard L -function generalises more easily to GSp_{2n} for general n . \diamond

6.3. Galois representations

Let \mathcal{F} be a genus 2 cuspidal Siegel modular form of weight $(3, 3)$ and level N which is an eigenform for the Hecke operators away from N . The following result shows that one can associate to \mathcal{F} a Galois representation.

THEOREM 15 (Weissauer, [Wei05]). *There exists a finite extension E of \mathbf{Q}_p and a 4-dimensional Galois representation $V_{\mathcal{F}}$ over E , such that for all primes ℓ coprime to pN we have*

$$\det(1 - X \mathrm{Frob}_\ell^{-1} \mid V_{\mathcal{F}}) = P_{\mathrm{spin}, \ell}(\mathcal{F}, X).$$

Perhaps surprisingly, these representations aren’t always irreducible, even if \mathcal{F} is cuspidal. This is because there are certain special types of cuspidal Siegel eigenforms that are “lifts” of automorphic forms on smaller groups; these are said to be *endoscopic*. There are several types of these, but only two which can occur in weight $(3, 3)$, namely *Yoshida lifts* and *Saito–Kurokawa lifts*.

If \mathcal{F} is non-endoscopic, and p is large enough¹ then the representation $V_{\mathcal{F}}$ is irreducible.

¹It’s expected to be irreducible for all p , but this is only known if we assume that $p \geq 5$ and $p \nmid N$.

THEOREM 16. *If \mathcal{F} is non-endoscopic, then $V_{\mathcal{F}}$ appears in the étale cohomology of the level N Siegel 3-fold. More precisely, we have a projection map*

$$\mathrm{Pr}_{\mathcal{F}} : H_{\text{ét}}^3 \left(\tilde{Y}_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(3) \right) \otimes E \longrightarrow V_{\mathcal{F}}^*.$$

We can similarly construct Galois representations for Siegel modular forms of weight (k_1, k_2) whenever $k_1 \geq k_2 \geq 3$, using étale cohomology with coefficients in sheaves coming from algebraic representations of the group GSp_4 .

REMARK. Note that weight $(2, 2)$ forms are not cohomological – they still have spin Galois representations, but these can't be seen in the cohomology of the Siegel threefold. This is unfortunate, since there is a conjecture due to Brumer and Kramer, the *Paramodular Conjecture*, predicting that (certain) abelian surfaces over \mathbf{Q} correspond to Siegel eigenforms of weight $(2, 2)$. It would be very interesting to try to build Euler systems for these non-cohomological eigenforms, by deforming the constructions of this chapter in a p -adic family. \diamond

6.4. Lemma-Flach elements

References: [Lem15, Lem17, LSZ17].

6.4a. Strategy. As we have seen above, the spin Galois representation of a genus 2 Siegel modular form can be found in the étale cohomology of the Siegel 3-fold $\tilde{Y}_1(N)$, for a suitable N . We therefore want to construct cohomology classes in $H_{\text{ét}}^4(\tilde{Y}_1(N) \times \mu_m^{\circ}, \mathbf{Z}_p(3))$ for $m \geq 1$, satisfying norm-compatibility relations as m changes (for a fixed N).

To do this, we note that we have a natural embedding

$$\iota : \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \longrightarrow \mathrm{GSp}_4,$$

which is given explicitly by

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right] \mapsto \begin{pmatrix} a & & & b \\ & a' & b' & \\ & c' & d' & \\ c & & & d \end{pmatrix}.$$

This embedding induces a map from the product of two modular curves into a Siegel 3-fold with compatible level structures; for instance, we get maps

$$Y_1(N) \times Y_1(N) \rightarrow \tilde{Y}_1(N),$$

which are injective if N is large enough. This in turn induces a pushforward map on the étale cohomology groups

$$\iota_* : H_{\text{ét}}^i(Y_1(N)^2, \mathbf{Z}_p(j)) \longrightarrow H^{i+2}(\tilde{Y}_1(N), \mathbf{Z}_p(j+1)).$$

Consider the case when $i = j = 2$: Then the exterior cup product of two Siegel units ${}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}$ defines an element of $H_{\text{ét}}^2(Y_1(N)^2, \mathbf{Z}_p(2))$, and we define the *Lemma-Flach element* for $m = 1$ to be

$${}_{c,d} \mathrm{LF}_{1,N} = \iota_*({}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}).$$

6.4b. Motivation: integral formulae. As in Chapter 3 above, there is a good motivation for why this class ${}_{c,d}\mathrm{LF}_{1,N}$ should be interesting.

There is an integral formula for the spin L -function of GSp_4 , which is due to Piatetskii-Shapiro. If \mathcal{F} is a non-endoscopic, holomorphic eigenform of weight $(3, 3)$ (or of any cohomological weight), then we can consider the integral

$$\int_{(\Gamma_1(N)\backslash\mathcal{H})^2} E(\tau_1, s)E(\tau_2, s)\mathcal{F}(\tau_1, \tau_2)dA$$

where $E(\tau, s)$ is a suitably-chosen family of Eisenstein series, and $dA = \frac{d\tau_1 \wedge d\bar{\tau}_1 \wedge d\tau_2 \wedge d\bar{\tau}_2}{\Im(\tau_1)\Im(\tau_2)}$ is the invariant measure on $(\Gamma_1(N)\backslash\mathcal{H})^2$. The general theory tells us that this unfolds into a product of local integrals, and the local integral at a finite place computes the spin L -factor.

The problem is that the local integral at ∞ is always zero! This can be fixed by replacing the holomorphic eigenform \mathcal{F} with an “evil twin” \mathcal{F}^g , which is a real-analytic but non-holomorphic function on $\tilde{Y}_1(N)(\mathbf{C})$ with the same Hecke eigenvalues as \mathcal{F} ; this doesn’t change the local integrals at the finite places, but gives us a non-vanishing archimedean integral.

As in the Rankin–Selberg setting, the Lemma–Flach class we’ve defined is naturally the image under the étale regulator of a motivic cohomology class. The main result of [Lem17] shows that the Beilinson regulator of this motivic class, paired with an appropriate differential on $\tilde{Y}_1(N)(\mathbf{C})$ coming from \mathcal{F}^g , gives Piatetskii-Shapiro’s integral for $L_{\mathrm{spin}}(\mathcal{F}, s)$ at $s = -\frac{1}{2}$.

REMARK. The g stands for “generic”. Representation-theoretically, the problem is that the discrete-series representations of $\mathrm{GSp}_4(\mathbf{R})$ come in pairs (“local L -packets”), consisting of a holomorphic representation and a non-holomorphic one, and it is the non-holomorphic one which is generic (admits a Whittaker model) and thus can contribute to the integral formula.

One can also replace $Y_1(N) \times Y_1(N)$ with a symmetric space associated to GL_2/K , where K is an imaginary quadratic field; this gives an alternative integral representation which does involve the holomorphic eigenform \mathcal{F} . However, it seems to be impossible to interpret this integral as the regulator of a motivic cohomology class, since the symmetric space for GL_2/K is not an algebraic variety. \diamond

6.4c. Lemma–Eisenstein classes. Our task is now to extend this to an Euler system: that is, to define classes ${}_{c,d}\mathrm{LF}_{m,N}$ for $m > 1$ satisfying good norm-compatibility properties. As before, we’ll start by defining classes on higher-level modular varieties, which are easier to work with, and proving norm-compatibility relations for these auxiliary classes.

Let us define $\mathcal{U}(M, N) = \left\{ \gamma \in \mathrm{GSp}_4(\hat{\mathbf{Z}}) : \gamma = \begin{pmatrix} I_2 & 0_2 \\ 0_2 & I_2 \end{pmatrix} \bmod \begin{pmatrix} M & M \\ N & N \end{pmatrix} \right\}$, and $\tilde{Y}(M, N)$ the corresponding Siegel threefold.

LEMMA 1. *If $M|N$, the group $\mathcal{U}(M, N)$ is normalised by the element $u = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$.*

Define $\iota_{M,N} : Y(M, N)^2 \rightarrow \tilde{Y}(M, N)$ to be the composite

$$Y(M, N)^2 \xrightarrow{\iota} \tilde{Y}(M, N) \xrightarrow{u} \tilde{Y}(M, N).$$

Here, $Y(M, N)^2$ denotes as above the fibre product of two copies of $Y(M, N)$ over μ_M° .

DEFINITION. *The Lemma-Eisenstein class ${}_{c,d}\text{LEis}_{M,N}$ is the image of ${}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}$ under $(\iota_{M,N})_*$. Here, we regard ${}_c g_{0,1/N} \sqcup {}_d g_{0,1/N}$ as an element of $H_{\text{ét}}^2(Y(M, N)^2, \mathbf{Z}_p(2))$ via pullback.*

6.4d. Norm relations. Exactly as before, one sees straightforwardly that the Lemma-Eisenstein classes satisfy norm relations as N changes.

PROPOSITION 9. *Suppose that $M|N$ and ℓ is a prime with $\ell | N$. Then*

$$(\text{pr}_1)_*({}_{c,d}\text{LEis}_{M,\ell N}) = {}_{c,d}\text{LEis}_{M,N},$$

where pr_1 is the natural quotient map $\tilde{Y}(M, N\ell) \rightarrow \tilde{Y}(M, N)$. □

(Exercise: formulate and prove a similar formula for $\ell \nmid N$.)

The second norm relation: changing M . Let us write τ_ℓ for the “non-standard” degeneracy map $\tilde{Y}(\ell M, N)$ to $\tilde{Y}(M, N)$, given by the right-translation action of $\begin{pmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \text{GSp}_4(\mathbf{Q}_\ell) \subset \text{GSp}_4(\mathbf{A}_f)$. Note that τ_ℓ factors as

$$\tilde{Y}(\ell M, N) \longrightarrow \tilde{Y}(M(\ell), N) \xrightarrow{\tilde{\pi}_{2,\ell}} \tilde{Y}(M, N),$$

where the first map is the natural degeneracy map.

THEOREM 17. *Suppose $\ell|M$ and $\ell M | N$. Then we have*

$$(\tau_\ell)_*({}_{c,d}\text{LEis}_{\ell M,N}) = \mathcal{U}'_\ell \cdot {}_{c,d}\text{LEis}_{M,N}.$$

Here, \mathcal{U}'_ℓ is the Hecke correspondence on $\tilde{Y}(M, N)$ given by the element of $\text{GSp}_4(\mathbf{A}_f)$ which is $\begin{pmatrix} \ell & & & \\ & \ell & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ at ℓ , and the identity elsewhere.

REMARK. Again, there is a similar but slightly more complicated formula in the case when $\ell \nmid M$ (but still $\ell M | N$). ◇

PROOF. We erect the following commutative diagram, in which all vertical arrows are the natural degeneracy maps:

$$\begin{array}{ccc}
Y(\ell M, N)^2 & \xrightarrow{\iota_{\ell M, N}} & \tilde{Y}(\ell M, N) \\
\parallel & & \downarrow \\
Y(\ell M, N)^2 & \longrightarrow & \tilde{Y}(M(\ell), N) \\
\downarrow & \diamond & \downarrow \tilde{\pi}_{1, \ell} \\
Y(M, N)^2 & \xrightarrow{\iota_{M, N}} & \tilde{Y}(M, N) \\
& & \searrow \tilde{\pi}_{2, \ell} \\
& & \tilde{Y}(M, N)
\end{array}$$

$\tilde{\pi}_{2, \ell}$ (diagonal arrow from $\tilde{Y}(\ell M, N)$ to $\tilde{Y}(M, N)$)

We claim that the middle arrow is actually injective. This is equivalent to the claim that

$$H(\mathbf{A}_f) \cap u\tilde{U}(M(\ell), N)u^{-1} = U(M\ell, N)^2,$$

which is an easy matrix computation.

When $\ell \mid M$, we see that the square marked \diamond has both horizontal arrows closed immersions, and both vertical arrows of degree ℓ^3 . So it is Cartesian, and we may conclude that the image of ${}_{c,d}\text{LEis}_{\ell M, N}$ under pushforward to $\tilde{Y}(M(\ell), N)$ coincides with the pullback of ${}_{c,d}\text{LEis}_{M, N}$. The result now follows by pushing both of these elements forward along the diagonal arrow and observing that $\mathcal{U}'_\ell = (\tilde{\pi}_{2, \ell})_* \circ (\tilde{\pi}_{1, \ell})^*$. \square

6.4e. Lemma-Flach classes and their norm relations. Let $m \geq 1$. We let ϖ_m denote the element of \mathbf{A}_f^\times whose ℓ -th component is $\ell^{v_\ell(m)}$. Then right translation by the element $\begin{pmatrix} \varpi_m & & & \\ & \varpi_m & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \in \text{GSp}_4(\mathbf{A}_f)$ induces a map

$$s_m : \tilde{Y}(m, mN) \rightarrow \tilde{Y}_1(N) \times \mu_m^\circ.$$

DEFINITION. We define the Lemma-Flach element ${}_{c,d}\text{LF}_{m, N}$ to be the image of ${}_{c,d}\text{LEis}_{m, mN}$ under $(s_m)_*$.

THEOREM 18. Let ℓ be prime such that $\ell \mid M$ and $\ell \mid N$. Then we have

$$\text{norm}_{\mathbf{Q}(\mu_m)}^{\mathbf{Q}(\mu_{\ell m})}({}_{c,d}\text{LF}_{\ell m, N}) = \mathcal{U}'_\ell \cdot {}_{c,d}\text{LF}_{m, N}.$$

PROOF. Analogous to the proof of the corresponding statement for Beilinson-Flach elements. \square

CHAPTER 7

Projects

7.1. An Euler system for $\mathrm{GSp}_4 \times \mathrm{GL}_2$

The goal of this project is to construct an Euler system for the Galois representations appearing in the cohomology of a certain 4-dimensional Shimura variety: namely, the product of a Siegel modular threefold and a modular curve.

The underlying Rankin–Selberg formula (in the sense of Section 3.3) was proven by Novodvorsky [Nov79]: it uses the embedding $\iota : \mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2 \hookrightarrow \mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ given by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \mapsto \left[\begin{pmatrix} a & & b \\ & a' & b' \\ & c' & d' \\ c & & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right].$$

Novodvorsky’s formula expresses values of the degree 8 L -function of an automorphic form \mathcal{F} for $\mathrm{GSp}_4 \times_{\mathrm{GL}_1} \mathrm{GL}_2$ in terms of an integral over $\mathrm{GL}_2 \times_{\mathrm{GL}_1} \mathrm{GL}_2$; roughly, one integrates $\iota^*(\varpi_{\mathcal{F}})$, where $\varpi_{\mathcal{F}}$ is a certain differential associated to \mathcal{F} , against an Eisenstein series on the first GL_2 factor of $\mathrm{GL}_2 \times \mathrm{GL}_2$.

The corresponding Euler system should roughly be constructed as follows: let Y be a modular curve, and let g_Y be a Siegel unit on Y . Its image under the Kummer map $\kappa_p(g_Y)$ is an element of $H_{\text{ét}}^1(Y, \mathbf{Q}_p(1))$. We can regard it as an element of $H_{\text{ét}}^1(Y \times Y, \mathbf{Q}_p(1))$ via pullback along the natural projection map $Y \times Y \rightarrow Y$ onto the first factor. Then the bottom class of the Euler system (with trivial coefficients) should be given by

$$\iota_*(\kappa_p(g_Y)) \in H_{\text{ét}}^5(S \times Y, \mathbf{Q}_p(3)),$$

where S denotes a Siegel 3-fold of a suitable level.

REMARK. As in Section 3.3, this is visibly the étale regulator of a motivic class, whose Beilinson regulator is a special value of Novodvorsky’s integral. \diamond

The aim of the project is to show that the methods of Chapters 4 and 6 can be adapted to give an Euler system of $\mathrm{GSp}_4 \times \mathrm{GL}_2$.

This project has several sub-projects, which are to some extent orthogonal (so it would be suitable for a fairly large team):

- Writing down the relevant cohomology classes over the cyclotomic extensions of \mathbf{Q} , using suitable level structures on Y and S .
- Proving norm-compatibility statements in the “vertical” aspect (i.e. proving a relation under the Galois corestriction map for the classes over $\mathbf{Q}(\mu_{p^{n+1}})$ and $\mathbf{Q}(\mu_{p^n})$ where p is a prime and $n \geq 1$). This ought to be

possible using a “Cartesian square” diagram analogous to those we had above for the Beilinson–Flach and Lemma–Flach elements.

- Proving the “horizontal” (or *tame*) norm relation (i.e. proving a relation under the Galois corestriction map for the classes over $\mathbf{Q}(\mu_{m\ell})$ and $\mathbf{Q}(\mu_m)$, where ℓ is a prime *not dividing* m). This is a rather more difficult problem, since extra Euler factors make an appearance, and we did not discuss it in these lecture notes. In [LLZ14] and [KLZ17], we proved these norm relations by explicit computations, but this method does not generalize easily. A different and much more flexible approach for proving the tame norm relations (using local zeta integrals) was developed in the GSp_4 -case in [LSZ17, §§3, 10], and this strategy should generalize¹ to $\mathrm{GSp}_4 \times \mathrm{GL}_2$.
- studying the case of higher weights (i.e. determining for which coefficient systems \mathcal{F} on $S \times Y$ one can construct elements in $H_{\text{ét}}^5(S \times Y, \mathcal{F}(3))$). The corresponding calculation for the GSp_4 -case can be found in [LSZ17, §§4, 8].

7.2. Euler systems and Selmer groups in Coleman families

Modular forms can often be deformed in p -adic families (Hida families of ordinary forms, or the more general finite-slope families constructed by Coleman [Col97]). One knows that the Euler system of Kato, and the Euler system of Beilinson–Flach elements, can be interpolated in Coleman families (see [Wan14], [Han15] for the former, and [LZ16] for the latter). This raises (at least) two natural questions.

- In the Beilinson–Flach setting, can one define a Selmer group attached to the family, and use the Euler system to give upper bounds on its size? There is a general approach to defining Selmer groups in families, due to Pottharst [Pot13], and it would be interesting to try to formulate and prove a bound for Pottharst’s Selmer groups in this setting.
- Are there analogous interpolation results for other Euler systems, e.g. in the Hilbert or Siegel settings? The existence of Hida families in these settings is well-known ([Hid89, TU99]), so as a first step one could show that the Euler systems vary in these families.

Andreatta, Iovita and Pilloni have recently shown the existence of p -adic families of finite-slope automorphic forms in both the Hilbert [AIP16] and the Siegel case [AIP15]; but whether the Euler systems interpolate in these families is an open question.

¹It is possible to give a new proof of the tame norm relations for Beilinson–Flach elements using local zeta integrals. This is currently being worked out by Giada Grossi.

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